B Web appendix

Proofs for Stress and Coping - An Economic Approach
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The web appendix provides simple proofs which are not included in the paper. It also provides more intermediate steps for some of the proofs in the paper.

B.1 The Bellman equation and optimality conditions

We start from the general specification of a Bellman equation in continuous time under uncertainty (see Wälde, 2012, ch. 11.1.2)

$$\rho V(W(t)) = \max_{m(t)} \left\{ u(c(t)) - v(m(t)) + \frac{1}{dt} E_t dV(W(t)) \right\}.$$

The differential of the value function reads, given the law of motion in (5) for $W(t)$ (see e.g. Wälde, 2012, ch. 10.2.3),

$$dV(W(t)) = V'(W(t)) \left\{ F\left(\frac{p(t)}{a(t)}, W(t)\right) - \delta(m(t), W(t)) \right\} dt$$
$$+ \{V(W(t) + G(g(t), W(t))) - V(W(t))\} dq(t).$$

Forming expectations E_t yields

$$dV(W(t)) = V'(W(t)) \left\{ F\left(\frac{p(t)}{a(t)}, W(t)\right) - \delta(m(t), W(t)) \right\} dt$$
$$+ E^g \{V(W(t) + G(g(t), W(t))) - V(W(t))\} \lambda dt,$$

where we assume that the jump size of $g(t)$ in (1) and the frequency of jumps of $q(t)$ are independent. Both sources of uncertainty are taken into account when the expectations operator E_t is applied. As the expected numbers of jump of $q(t)$ over a small time interval dt is given by λdt, only the expectations with respect to $g(t)$ need to be formed. They are denoted by E^h as this expectation refers to the random size of $h(t)$ in (1). Dividing by dt and replacing this expression in the above general specification yields

$$\rho V(W(t)) = \max_{m(t)} \left\{ \begin{array}{c} u(c(t)) - v(m(t)) + V'(W(t)) \left[F\left(\frac{p(t)}{a(t)}, W(t)\right) - \delta(m(t), W(t)) \right] \\ +\lambda \left[E^h V(W(t) + G(g(t), W(t))) - V(W(t)) \right] \end{array} \right\}. \tag{B.1}$$

This is the Bellman equation for the case where $W(t) < \bar{W}$.

The second equation we need to take into account is value matching condition

$$V(W(t)) = V(W(t) - \Delta) - v^M \text{ for } W(t) \geq \bar{W}. \tag{B.2}$$

It simply says that the value of optimal behaviour $V(W(t))$ for tension beyond the threshold level is given by the value to which tension jumps after intervention, $V(W(t) - \Delta)$, minus the immediate costs v^M. The jump in tension by Δ reflects the change in (4) due to an emotional outburst.

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The first-order condition for $m(t)$ in (B.1) requires

$$v'(m(t)) = -V'(W(t)) \frac{\partial}{\partial m(t)} \delta(m(t), W(t)). \tag{B.3}$$

Assuming an interior solution, marginal costs from coping, $v'(m(t))$, must equal marginal gains, $-V'(W(t)) \frac{\partial}{\partial m(t)} \delta(m(t), W(t))$. As stress $W(t)$ is "a bad" and not a good, $V'(W(t))$ is negative and the right-hand side of this first-order condition indeed reflects a positive marginal gain. The first-order condition implicitly makes optimal coping a function of tension, $m(t) = m(W(t))$.

If we also want to endogenize the threshold level $\bar{W}$ for outbursts, we would require $dV(\bar{W})/d\bar{W} = 0$. If $\bar{W}$ is left exogenous, the condition in (B.3) is the only first-order condition.

B.2 Functional forms and their implications

B.2.1 Closed-form solution for the individual case

The functional forms are given in (8). The value matching condition reads as in (B.2).

- The Bellman equation and first-order condition

With these functional forms and without time arguments, our Bellman equation (B.1) reads

$$\rho V(W) = \max_m \left\{ \begin{array}{c} u(c, W) - v_0 m^{1+\zeta} + V'(W)[\Phi W - \delta_1 m] \\ +\lambda \left[E^h V(W - \chi g) - V(W)\right] \end{array} \right\}. \tag{B.4}$$

The first-order condition becomes

$$(1+\zeta) v_0 m^\zeta = -V'(W) \delta_1. \tag{B.5}$$

- Guess of value function

We start with a guess for the value function $V(W)$, which we denote by $J(W)$,

$$\begin{aligned} J(W) &= \Lambda_0 - \Lambda_1 W, \\ J'(W) &= -\Lambda_1. \end{aligned} \tag{B.6}$$

We have two free parameters, Λ_0 and Λ_1.

- Verification

We need to verify that this guess satisfies the value-matching condition, the first-order condition and the Bellman equation. The value-matching condition (B.2) is satisfied if

$$\Lambda_0 - \Lambda_1 W = \Lambda_0 - \Lambda_1 [W - \Delta] - v^M \Leftrightarrow v^M = \Lambda_1 \Delta \Leftrightarrow \Lambda_1 = \frac{v^M}{\Delta}. \tag{B.7}$$

This fixes the first parameter Λ_1.

The first-order condition (B.5) is satisfied if

$$(1+\zeta) v_0 m^\zeta = \Lambda_1 \delta_1 = \delta_1 \frac{v^M}{\Delta}$$

where the second equality used (B.7). This fixes a constant coping level m at

$$m^\zeta = \frac{v^M}{\Delta} \frac{\delta_1}{(1+\zeta) v_0} \Leftrightarrow m = \left(\frac{v^M}{\Delta} \frac{\delta_1}{v_0} \frac{1}{1+\zeta} \right)^{1/\zeta}. \tag{B.8}$$

Finally, the Bellman equation (B.4) with $u(c, W)$ from (8f) and consumption from (8e) holds if

$$\begin{aligned}
&(\rho+\lambda)(\Lambda_0-\Lambda_1 W) \\
&= \nu w[M-\kappa W]-\alpha W-v_0 m^{1+\zeta}-\Lambda_1[\Phi W-\delta_1 m]+\lambda E^h[\Lambda_0-\Lambda_1[W-\chi g]] \\
&= \nu w M-(\nu w\kappa+\alpha)W-v_0 m^{1+\zeta}-\Lambda_1[\Phi W-\delta_1 m]+\lambda E^h[\Lambda_0-\Lambda_1[W-\chi g]] \\
&\equiv \nu w M-\Psi W-v_0 m^{1+\zeta}-\Lambda_1[\Phi W-\delta_1 m]+\lambda E^h[\Lambda_0-\Lambda_1[W-\chi g]],
\end{aligned}$$

where the last equality defined

$$\Psi \equiv \nu w\kappa+\alpha. \tag{B.9}$$

Now we collect constant terms and terms proportional to W,

$$\begin{aligned}
(\rho+\lambda)(\Lambda_0-\Lambda_1 W) = \nu w M-\Psi W-v_0 m^{1+\zeta}-\Lambda_1[\Phi W-\delta_1 m] \\
+\lambda E^h[\Lambda_0-\Lambda_1[W-\chi g]],
\end{aligned}$$

$$\begin{aligned}
(\rho+\lambda)\Lambda_0-(\rho+\lambda)\Lambda_1 W-\nu w M = -\Psi W-v_0 m^{1+\zeta}-\Lambda_1\Phi W+\Lambda_1\delta_1 m \\
+\lambda\Lambda_0-\lambda\Lambda_1 W+\lambda\Lambda_1\chi E^h g
\end{aligned}$$

Now cancel first $\lambda\Lambda_0-\lambda\Lambda_1 W$ on both sides to find

$$\rho\Lambda_0-\rho\Lambda_1 W-\nu w M = -\Psi W-v_0 m^{1+\zeta}-\Lambda_1\Phi W+\Lambda_1\delta_1 m+\lambda\Lambda_1\chi E^h g \Leftrightarrow$$

$$\rho\Lambda_0-\nu w M+v_0 m^{1+\zeta}-\Lambda_1\delta_1 m-\lambda\Lambda_1\chi E^h g = ((\rho-\Phi)\Lambda_1-\Psi)W.$$

Rearranging further and using (B.7), we get

$$\rho\Lambda_0-\nu w M+v_0 m^{1+\zeta}-\frac{v^M}{\Delta}\left[\delta_1 m+\lambda\chi E^h g\right] = \left((\rho-\Phi)\frac{v^M}{\Delta}-\Psi\right)W.$$

This holds if two conditions are fulfilled simultaneously. The first makes sure that the right-hand side equals zero. The second makes sure that the left-hand side equals zero. The first reads with (12) and with (B.9)

$$\rho = \Phi+\Psi\frac{\Delta}{v^M} = \phi\frac{p}{a}-\delta_0+(\nu w\kappa+\alpha)\frac{\Delta}{v^M}. \tag{B.10}$$

which is (9) in the main text. The second pins down Λ_0 from our guess which needs to satisfy

$$\rho\Lambda_0 = \nu w M-v_0 m^{1+\zeta}+\frac{v^M}{\Delta}\left[\delta_1 m+\lambda\chi E^h g\right] \tag{B.11}$$

where the surprise g is from (1). This is used in prop. 5 in the main text.

- The flexibility of the parameter restriction

Can we allow the slope in (B.13), i.e. $\phi\frac{p}{a}-\delta_0$, to take positive and negative values? The restriction on this slope comes from (B.10) where we need the right-hand side to be positive as otherwise ρ would not be positive. When the slope $\phi\frac{p}{a}-\delta_0$ is positive, ρ in (B.10) is positive as well. When the slope is negative, (B.10) requires

$$\phi\frac{p}{a}-\delta_0+\Psi\frac{\Delta}{v^M} > 0, \tag{B.12}$$

i.e. $\Psi\frac{\Delta}{v^M}$ must be sufficiently large.

B.2.2 The optimal stress level

Tension follows (5),

$$dW(t) = \left\{ F\left(\frac{p(t)}{a(t)}, W(t)\right) - \delta(m(t), W(t)) \right\} dt + G(g(t), W(t))\, dq(t).$$

Given the functional forms in (8) and (1) and optimal consumption from (10), this reads

$$dW(t) = \{\Phi W - \delta_1 m\} dt - \chi[h(t) - \mu]\, dq(t)$$

where Φ is from (12). The evolution of $W(t)$ is subject to (4), i.e. subject to $W(\tau_i) = W(\tau_{i-}) - \Delta$ for $W > \bar{W}$.

Assuming there are no surprises, we get

$$\dot{W}(t) = \Phi W(t) - \delta_1 m \tag{B.13}$$

and can solve this equation for $W < \bar{W}$, i.e. for the range where there are no outbursts,

$$W(\tau) = W(t)\, e^{\Phi[\tau - t]} - \delta_1 m \int_t^\tau e^{\Phi[\tau - s]} ds = W(t)\, e^{\Phi[\tau - t]} - \frac{\delta_1 m}{\Phi}\left[e^{\Phi[\tau - s]}\right]_t^\tau$$

$$= W(t)\, e^{\Phi[\tau - t]} - \frac{\delta_1 m}{\Phi}\left[1 - e^{\Phi[\tau - t]}\right] = \left(W(t) + \frac{\delta_1 m}{\Phi}\right) e^{\Phi(\tau - t)} - \frac{\delta_1 m}{\Phi}$$
$$= (W(t) - W^*)\, e^{\Phi(\tau - t)} + W^*$$

where W^* is from (13).

B.3 References (for web appendix)

Wälde, K., 2012, Applied Intertemporal Optimization. Know Thyself - Academic Publishers, available at www.waelde.com/KTAP