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The economic determinants of technology shocks in a real business cycle model

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Abstract

Creative destruction due to new technologies causes both long-run growth and short-run business fluctuations. New technologies push the economy on a higher productivity level but also imply partial obsolescence of old production units. Obsolescence induces an adjustment period during which growth rates gradually fall. At some endogenously determined point, research for the next technology starts. Once the new technology is discovered, a technology shock occurs and the next cycle starts. This is shown in a continuous time RBC model with risk-averse agents where savings can be used for financing deterministic capital accumulation and stochastic R&D for new technologies. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The most visible properties of most macroeconomic time series are their positive long-run growth trend and their short-run fluctuations. It seems to be widely accepted that the rate of growth of an economy is a function

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of technologies, institutional arrangements and other characteristics of the economy under consideration. There is considerable dispute, however, to what extent short-run fluctuations are endogenous as well. One could hold the view that fluctuations in growth rates are largely induced by oil price shocks or other exogenous events.

The present paper argues that both the growth rate of an economy and its technology-shock-induced fluctuations are endogenous phenomena. To illustrate this point of view, we consider an economy where savings can be allocated to capital accumulation (as in the textbook Ramsey growth model) or to financing R&D (as in Aghion and Howitt, 1992). Capital accumulation is a smooth and deterministic process while R&D is uncertain and takes place in discrete steps. [2] Whenever R&D is successful, total factor productivity increases—a technology shock occurs. As the amount of resources invested into R&D depends on relative returns, the occurrence of technology shocks is determined by economic incentives: When capital is scarce relative to the current technology level, resources will be allocated to capital accumulation. When capital is relatively abundant, financing R&D is more profitable. As relative returns depend on relative abundance, they continuously change and the economy permanently shifts investments back and forth between investment and R&D. On average, there is long-run positive growth but growth rates are fluctuating.

A typical cycle of our economy will be as follows. Starting with some capital level, all resources will initially be allocated to capital accumulation. Decreasing returns to capital imply that capital accumulation loses in profitability relative to financing R&D. At some point of time, expected returns from financing R&D exceed certain returns from allocating capital and all investment will be allocated to R&D. For some time, research will not imply development but at some uncertain point of time, a new technology will have been found, total factor productivity rises and the economy starts accumulating capital again.

The present paper can be seen as an extension of the real business cycle approach (Kydland and Prescott (1982); an overview is provided by Stadler (1994) and Cooley (1995); recent contributions include Hairault et al. (1997); Canova and Ubide (1998) and Weder (2000)). It presents a model that allows to study economic determinants of technology shocks and how new technologies gradually spread in an economy. Technology shocks will be understood here in a narrow sense as shocks that are caused by successful innovations in production or management practices. The point of time when a technology shock takes place will be determined by economic trade-offs and the propagation mechanism is a natural consequence of technological progress.

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[2] Shell (1966) argues as well that capital accumulation and R&D differ in the degree of riskiness in the way modelled here.

In addition to studying the economic determinants of technology shocks, this approach adds two further insights to the RBC literature. First, a very intuitive propagation mechanism is provided which is related to the literature on embodied technological change (of which recent examples are Boucekkine et al., 1997a,b). Once a technology shock occurs, only the acquisition of new capital goods will be financed. A technology shock is therefore permanent. (For alternative approaches cf. e.g. Perli, 1998 or Collard, 1999.)

In order to make a comparison of results with standard RBC models as straightforward as possible, these mechanisms will be studied in a centrally planned economy. Once the economy is decentralized, a second interesting result will become available. The property of many RBC models that decentralized economies are socially optimal no longer holds when technology shocks are endogenous, simply because the economic activities which cause technology shocks are subject to various market failures. While this result would be expected if innovation was embedded in an imperfect competition setup, it holds here as well, despite price-taking behaviour and zero profits on all markets. The market failure results from individual investment decisions into R&D in a quality-ladder setup à la Grossman and Helpman (1991a) and Aghion and Howitt (1992). Individuals invest into R&D as they will receive interest payments for the new production unit resulting from R&D. They neglect, however, social gains from higher total factor productivity for all subsequent capital goods.

This last aspect points to another interesting result of the present paper: Individual investment into R&D in a decentralized economy under perfect competition does not necessarily have to be zero. Successful R&D as modelled here does not only lead to new technologies that have a higher total factor productivity but also to a prototype of the new production unit. Returns to R&D can therefore at least partially be privately appropriated, even under perfect competition. Investment into R&D will be covered by future payments to owners of the prototype.

There is a small literature that jointly studies endogenous growth and endogenous technology shocks.[3] Bental and Peled (1996) present a model that shares some equilibrium properties of the model presented below. They focus on specific utility and production functions (logarithmic and Cobb–Douglas), however, and analyse their effects in an discrete time 2-period OLG framework (that implies a fixed saving rate). Jovanovic and Rob (1990) study the endogenous determination of major discoveries and subsequent refinement but they view their analysis more in an industry rather than an economy wide

[3] Several papers studied the effect of technological innovations on economic activity. This literature takes the arrival of new technologies as exogenous event, however. See e.g. Cheng and Dinopoulos (1996), Chou and Shy (1993), Helpman and Trajtenberg (1998), Jones and Newman (1995). Schleifer (1986) showed in his classic paper how inventions that occur non-cyclically turn into cyclical innovations.

context. Further, the modelling approach chosen here is more closely linked to standard RBC models. Matsuyama's (1999b) analysis predicts that an economy either stagnates, grows with a constant positive growth rate or grows in cycles, depending on parameter values including productivity parameters and the (exogenous) saving rate (cf. Matsuyama, 1999a however). In his model, an innovation depletes the capital stock and makes innovations (denominated in units of capital) more costly. Incentives for innovations result from a one-period monopoly enjoyed by the innovator. Here, an innovation raises the returns to capital accumulation by increasing total factor productivity and therefore delays further innovations. There are incentives for innovations (even in a decentralized economy) because a successful R&D process leads to a new production unit that is owned by those having financed R&D. [4]

Formally, the present model differs from these approaches in its continuous time setup which makes the model more tractable. In addition, we place our analysis in a standard Ramsey growth model context with general production and utility functions. The main contribution from a technical perspective is the explicit analysis of optimal behavior of risk averse agents in a fully specified general equilibrium model with Poisson uncertainty. From this perspective, the paper is part of the continuous time and uncertainty literature, going back to Merton (1969). Recent contributions include Dixit and Pindyck (1994), Farzin et al. (1998), Turnovsky (1995, part IV), Turnovsky (1999), and Wälde (1999a, b).

The next section presents the central planner economy. Section 3 analyses the equilibrium growth path exhibiting endogenous cycles. Section 4 decentralizes the economy and Section 5 concludes. Appendix A contains various proofs.

2. The model

2.1. The economy

The economy is endowed with capital K and labour L. It produces a consumption good C, an investment good I and it undertakes R&D. Technolo-

[4] Other papers study endogenous growth and fluctuations that do not result from technology shocks (Aghion and Howitt, 1992, 1998, Chapter 8; Deissenberg and Nyssen, 1998) and are therefore not as closely linked to the RBC approach as the present paper. Endogenous cycles are also generated through optimal scrapping rules leading to delay differential equations (Boucekkine et al., 1999) and through time-to-build mechanisms (Wen, 1998). In the sunspot literature (e.g. Drugeon and Wigniolle, 1996) cycles occur because of exogenous events (like sunspots) which change beliefs of individuals and, through the multiplicity of equilibria, factor allocation. A more detailed overview of various approaches to business fluctuations is provided in Wälde (1998, Chapter 1).

gies for these production activities differ only in their total factor productivity (TFP). TFP is constant in the investment-good and in the R&D sector. In the consumption-good sector, TEP is given by A^{γ} and is subject to discontinuous technological progress.[5] Each new technology increases γ by one and total factor productivity by $A > 1$.[6] The technology $G(.)$ has positive first and negative second partial derivatives. As it is also characterized by constant returns to scale, the economy's resource constraint can be written as

$$A^{-\gamma}C + I + R_{\mathrm{R\&D}} = G(K,L), \tag{1}$$

where $R_{\mathrm{R\&D}}$ are resources allocated to R&D.

Allowing for deterministic exponential capital depreciation at rate δ and given that the production of the investment good per unit of time $\mathrm{d}t$ equals deterministic capital accumulation $\mathrm{d}K^d$ plus depreciation,

$$I\,\mathrm{d}t = \mathrm{d}K^d + \delta K\,\mathrm{d}t, \tag{2}$$

we obtain $\mathrm{d}K^d = (G(K,L) - \delta K - R_{\mathrm{R\&D}} - A^{-\gamma}C)\,\mathrm{d}t$. The capital stock increases in a deterministic fashion if the difference between (a measure of) aggregate output $G(K,L)$, depreciation δK, investment in R&D $R_{\mathrm{R\&D}}$ and productivity adjusted consumption $A^{-\gamma}C$ is positive. Denoting for simplicity

$$F(.) \equiv G(K,L) - \delta K \tag{3}$$

in what follows, we have

$$\mathrm{d}K^d = (F(K,L) - R_{\mathrm{R\&D}} - A^{-\gamma}C)\,\mathrm{d}t. \tag{4}$$

R&D is directed at developing new production units. When a new production unit has been developed, it represents a capital stock of $\varpi \geqslant 0$ whose total factor productivity exceeds total factor productivity of the previous vintage by A. It is further assumed that after a successful development of a new technology, a certain share $1 - s$ of the previous vintage can be upgraded which therefore also has the higher total factor productivity. The remaining capital stock (which might be zero) becomes obsolete. R&D therefore implies a creative destruction mechanism á la Schumpeter. Denoting by $\tilde{K}$ the capital stock after successfully finishing an R&D project and by K the current aggregate capital stock, we have

$$\tilde{K} = \varpi + (1 - s)K. \tag{5}$$

[5] This discontinuous technological progress mechanism and the R&D technology presented below are borrowed from Aghion and Howitt (1992). It has been used in a similar way by Grossman and Helpman (1991a) and others.

[6] Restricting TFP growth to the consumption good sector considerably simplifies presentation of results and was chosen for this reason only. Some additional insights can be obtained, and will briefly be discussed in the conclusion, when all sectors are subjected to technological progress. This does not justify increased complexity as results are qualitatively identical (Wälde, 1998, Chapter 4).

By this formulation,[7] at each instant in time, the entire capital stock is of one unique vintage.[8]

The crucial difference between R&D and capital accumulation is uncertainty associated with R&D; *research* for the next technology does not necessarily lead to *development* of the next technology. When searching for a new technology, new paths have to be explored which have not been explored by others before. This uncertainty is captured by a Poisson process whose arrival rate λ is given by

$$\lambda = bR_{\mathrm{R\&D}}, \tag{6}$$

where TFP is denoted by the constant b. Allocating $R_{\mathrm{R\&D}}$ to research implies that the probability per unit of time $\mathrm{d}t$ that research indeed leads to development of a new production unit is given by $\lambda\,\mathrm{d}t$ while the probability that research remains without success is given by $1-\lambda\,\mathrm{d}t$.

Defining finally θ as the share of net savings allocated to R&D,[9]

$$R_{\mathrm{R\&D}} = \theta[F(.) - A^{-\gamma}C],$$

we can conveniently summarize the economy's consumption and investment opportunities as

$$\mathrm{d}K = (1-\theta)(F(.) - A^{-\gamma}C)\,\mathrm{d}t + (\tilde{K} - K)\,\mathrm{d}q. \tag{7}$$

This is a stochastic differential equation where uncertainty results from a Poisson process q. During a small period of time $\mathrm{d}t$, the capital stock of vintage γ increases deterministically by the share $1-\theta$ of total savings $F(.) - A^{-\gamma}C$ allocated to capital accumulation. When R&D is undertaken, the capital stock is also subject to abrupt changes. With q denoting the Poisson process resulting from investment in R&D projects, $\mathrm{d}q$ is the increment of this process. A successful R&D project implies $\mathrm{d}q = 1$. The capital stock then changes by $\tilde{K} - K$ and productivity in the consumption good sector rises by A. When no investment in R&D takes place or when R&D fails, the increment is zero, $\mathrm{d}q = 0$.

[7] New technologies are therefore partly vintage specific, partly disembodied. Both views have their empirical merit. Hulten (1992) finds that 20% of total factor productivity growth stems from embodied technological change. Greenwood et al. (1997) argue that up to 60% of total factor productivity growth stems from embodied technological change.

[8] The specific process and economic mechanisms of upgrading and obsolescence is left in the background. One could imagine an endogenous obsolescence process similar to Gilchrist and Williams (1998) where capital goods with a productivity above a certain threshold level are used, while those below this level are left idle. An alternative approach is followed by Boucekkine et al. (1999), who show how optimal replacement decisions are related to endogenous business fluctuations.

[9] In the case of ambiguity, parantheses include arguments of functions; squared brackets always indicate a multiplication operation.

Note that the arrival rate is linear in net investment $\theta[F(.) - A^{-\gamma}C]$ in R&D. This assumption is necessary if one wants to understand the R&D sector (in a decentralized economy) as a multitude of firms that produce under perfect competition. Increasing or decreasing returns in the arrival rate (just as increasing or decreasing returns to scale) are not easily reconciled with perfect competition. This linearity will be crucial for the results derived below but, in the light of this perfect competition requirement, is a natural assumption.

2.2. The central planner's choice

2.2.1. A bang-bang result

The planner's objective is to maximize a social welfare function given the above technologies. Letting the value of the optimal program at t be denoted by $V(K,\gamma)$, her objective is

$$V(K,\gamma) = \max_{\{C(\tau),\,\theta(\tau)\}} \mathrm{E} \int_t^{\infty} \mathrm{e}^{-\rho[\tau-t]} u(C(\tau))\,\mathrm{d}\tau \tag{8}$$

subject to (5) and (7). The planner maximizes expected utility from discounted consumption flows by choosing a stream of consumption $\{C(\tau)\}$ and allocation $\{\theta(\tau)\}$ of savings to financing R&D and capital accumulation. The expectations operator is E and ρ denotes the time preference rate. The share $\theta(\tau)$ is constrained to lie between zero and unity where $\theta = 1$ means that all savings are allocated to R&D.[10] The instantaneous utility function is increasing in consumption with decreasing slope,

$$u'(C) > 0, \quad u''(C) < 0. \tag{9}$$

The planner's maximization problem can be expressed by the Bellman equation (cf. e.g. Dixit and Pindyck, 1994)

$$\begin{aligned}\rho V(K,\gamma) = \max_{C,\theta}\{&u(C) + V_K(K,\gamma)\,[1-\theta][F(.) - A^{-\gamma}C] \\ &+ \lambda(\theta[F(.) - A^{-\gamma}C])\,[V(\tilde{K},\gamma+1) - V(K,\gamma)]\}.\end{aligned} \tag{10}$$

The expression to be maximized is given by the sum of instantaneous utility $u(C)$, the value of additional units of capital and the expected value of a new technology. The value of additional units of capital is given by the marginal value of the current stock of capital $V_K(K,\gamma)$ times the share $1-\theta$ of savings

[10] If we allowed for $\theta > 1$, we would technologically allow capital to be decumulated, retransformed into the aggregate good which is then used for the R&D process. While standard in models of optimal growth where capital can be 'eaten up' we exclude this by assumption. On equilibrium paths as studied below the planner would never want to set θ above unity. A negative θ would imply negative resource allocation to the R&D process. This is clearly technologically unfeasible.

allocated to accumulation of physical capital times savings $F(.) - A^{-\gamma}C$. The expected value of a new technology equals the product of the arrival rate of new technologies (6) and the gain from holding a capital stock $\tilde{K}$ suitable for production with the next technology $\gamma + 1$ as compared to the value of holding the current capital stock under technology γ.

The first order condition for consumption can be written after dropping the vintage arguments [11] as

$$u'(C) = [1 - \theta]A^{-\gamma}V'(K) + \theta b A^{-\gamma}[V(\tilde{K}) - V(K)]. \tag{11}$$

Marginal utility from consumption today is the θ-weighted sum of marginal utility from consumption in the future. Higher future consumption either comes from a higher capital stock or a better technology. The value of more capital through less consumption is given by the value of an additional unit of capital, $V'(K)$, times the instantaneous increase in the capital stock by reducing consumption by one unit $[1 - \theta]A^{-\gamma}$. The impact of less current consumption on future technologies is captured by the increase in the probability of a research success $\theta b A^{-\gamma}$ times the gain in the value of the optimal program if research is indeed successful, $V(\tilde{K}) - V(K)$.

The derivative of (10) with respect to the share θ invested in R&D,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\{.\} = (-V'(K) + b[V(\tilde{K}) - V(K)])\,[F(.) - A^{-\gamma}C], \tag{12}$$

is a function independent of the share θ itself. Therefore, except for some K^*, where the planner is indifferent, $\theta = 0$ or $\theta = 1$. Assuming that net savings are positive, [12] $F(.) > A^{-\gamma}C$,

$$\left.\begin{array}{l} \theta = 0 \\ \theta \in [0,1] \\ \theta = 1 \end{array}\right\} \Leftrightarrow b[V(\tilde{K}) - V(K)] \left\{\begin{array}{c} < \\ = \\ > \end{array}\right\} V'(K). \tag{13}$$

The planner allocates all net savings to capital accumulation ($\theta = 0$) when the marginal value of capital exceeds the discrete increase of the value of capital times the marginal arrival rate, while all savings net of capital depreciation are allocated to R&D ($\theta = 1$) when the derivative (12) is positive. Depreciation of capital at rate δ implies that some production of investment goods always takes place. When $\theta = 1$, net capital accumulation, which by (7) equals $\mathrm{d}K = (1 - \theta)(F(K,L) - A^{-\gamma}C)\,\mathrm{d}t$, is zero. By (2), however, $I = \delta K$.

This complete specialization is a surprising result, at first sight. The portfolio choice literature says that an individual invests always at least some

[11] No ambiguity arises as K always denotes the current capital stock and $\tilde{K}$ denotes the capital stock available for the next vintage. We then also use $V'(K)$ as an abbreviation for $V_K(K, \gamma)$.

[12] We restrict attention to this case. An extension is straightforward.

amount of her wealth into the risky asset. Here, there are three reasons for this bang-bang property. First, the paper studies financing risky R&D and not financing risky investment. Risky investment is characterized by payoffs that are proportional to investment where the factor of proportionality is the uncertain return. The higher investment, the higher the expected payoff. By contrast, risky R&D is characterized by fixed payoffs and incentives to invest in R&D at all stem from the impact of investment on the arrival rate. This is a property of any R&D project, both in this central planner framework and in a decentralized setup.

Second, this bang-bang property is a result of the linearity of the arrival rate in θ. As mentioned before, this is a standard assumption which is required here if one wants a technology for R&D that can be replicated by decentralized competition between many firms.

A third, technical, reason for this bang-bang behaviour is the continuous time framework used here. In discrete time, an interior solution results both from risky investment and risky R&D, as was shown in Wälde (1998, Chapter 8). This paper also shows that R&D is characterized by fixed payoffs when R&D is financed by many individuals in a decentralized economy.[13]

It is intuitively clear what would happen if any of these conditions was not satisfied. Condition (13) would hold as an equality for some range of the capital stock and not only for a specific K^*. More importantly, the share θ of savings allocated to R&D would increase, the higher the economy's capital stock: As returns to investment in capital accumulation fall, more and more resources would be shifted towards investing in R&D. Hence, the probability per unit of time that a new technology is found would increase in the economy's capital stock. Eventually, when a new technology is found, the share θ would fall again.[14] Hence, qualitatively, this shift from investing in capital accumulation to investing in R&D takes place more smoothly but in a similar way. The strong prediction of the model should therefore be seen as a useful device that simplifies the derivation of results and not as a one-to-one mapping of reality.

2.2.2. Deterministic regime

When the share of savings invested into R&D are zero, $\theta = 0$, the first order condition for consumption (11) simplifies to

$$u'(C) = A^{-\gamma} V'(K). \tag{14}$$

The optimal consumption level is reached when marginal utility from consumption equals the marginal value of capital. This condition is frequently

[13] If the utility function was not additive over time, an interior solution might also obtain.

[14] Such a prediction can be shown to follow from a model with uncertain investment rather than uncertain R&D.

found in deterministic setups where it usually appears in a form like $u'(C) = \mu$, where μ is the Hamiltonian multiplier which stands for the marginal valuation of an additional unit of the associated state variable (cf. e.g. Kamien and Schwartz, 1991), just as here. The Keynes–Ramsey rule is the familiar one,

$$-\frac{u''(C)}{u'(C)}\,\mathrm{d}C = [F_K(.) - \rho]\,\mathrm{d}t, \tag{15}$$

and the resource constraint (7) becomes

$$\mathrm{d}K = [F(.) - A^{-\gamma}C]\,\mathrm{d}t. \tag{16}$$

2.2.3. Switch to the stochastic regime

As long as gains from an additional unit of capital exceed expected gains from a new technology, the economy is in the deterministic regime. When the share derivative (12) holds with equality, the planner switches from $\theta = 0$ to 1. In its current formulation, however, condition (12) is of little use since the value function is not known. Assuming that the economy starts with a capital stock K_0 that implies that capital is accumulated, i.e. $\theta = 0$,[15] one can express this condition in a form that is more intuitive and that can be used for a simple phase diagram analysis. Defining

$$\Omega \equiv Au'(C(\tilde{K}))/u'(C(K)) \tag{17}$$

as the ratio of marginal utility after the development of a new technology when the capital stock will be given by $\tilde{K}$ and marginal utility at current consumption, the Appendix proves the following

Theorem 1 (Optimal investment decision).

$$\theta = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Leftrightarrow F_K \begin{Bmatrix} > \\ = \end{Bmatrix} \rho + \lambda[1 - [1 - s]\Omega]. \tag{18}$$

The planner allocates all savings to deterministic capital accumulation ($\theta = 0$) when the marginal product of capital is sufficiently high. Sufficiently high in this context means that it must be higher than the time preference rate *plus* the (adjusted) arrival rate where the arrival rate (6) is evaluated at current output and consumption. The planner is willing to accumulate an additional unit of capital, i.e. $\theta = 0$, when the return to saving capital exceeds the time preference rate plus the (adjusted) arrival rate. More capital increases utility of the planner only if she is compensated for her impatience and for the risk that this capital stock eventually becomes obsolete.

[15] The economy could start with a capital stock that implies that the planner allocates all savings to R&D. Once a new technology has been developed, we are back to $\theta = 0$.

The degree to which the arrival rate is adjusted depends on the share s of capital that becomes useless under the new technology. When the entire capital stock is useless under the new technology, i.e. $s = 1$, adjustment for risk is given by the arrival rate λ. The interest rate which must be guaranteed when accumulating more capital is then given by $\rho + \lambda$. The sum of the time preference rate and the arrival rate is known to be the discount factor that is used by households when discounting an income stream that ends at some exponentially distributed point in time, where λ is the parameter of the exponential distribution. [16]

When the arrival of a new technology does not imply that the entire old capital stock becomes obsolete, i.e. when $s < 1$, the arrival rate is adjusted by the instantaneous change in marginal utility Ω due to the new technology. Depending on the sign of $1-[1-s]\Omega$, the economy stops accumulating capital at an interest rate that is higher or lower than the time preference rate. When s is sufficiently close to unity, the adjusted arrival rate is close to the arrival rate. As the obsolescence parameter s falls, and with Ω sufficiently high, [17] $1 - [1 - s]\Omega$ becomes negative and the arrival of new technologies induces individual to accumulate more capital than in a purely deterministic world.

A further property of the switch from the deterministic to the stochastic regime is that no consumption jump takes place. [18]

2.2.4. Stochastic regime

When all net investment is channelled into R&D, $\theta = 1$, and the planner is indifferent, we know from the above theorem (18) that marginal productivity of capital equals the sum of the time preference rate and the adjusted arrival rate. The economy is in a (transitory) stationary state. The capital stock is determined by history, i.e. equals the stock of capital at the moment the economy allocates all savings to R&D. As depreciation continues, however, some investment goods are still produced, $I = \delta K$, which follows from (2).

The consumption level at every moment in time is determined by the consumption first order condition (11). In the stochastic regime where $\theta = 1$, this condition reads

$$u'(C) = bA^{-\gamma}[V(\tilde{K}) - V(K)]. \tag{19}$$

[16] Compare e.g. Blanchard (1985), Grossman and Helpman (1991b) or Aghion and Howitt (1992) among others.

[17] Note that Ω is endogenous and therefore not independent of s. In equilibrium, however, Ω is constant.

[18] Under the deterministic regime, the marginal value of the capital stock equals marginal utility from consumption (13). On the R&D line, where the derivative with respect to the share θ in (11) is zero, this holds as well. This can be seen by inserting $\mathrm{d}\{.\}/\mathrm{d}\theta = 0$ from (11) into the consumption first order condition (10). Since for both the R&D line and the deterministic regime $u'(C) = V'(K)$ and since the capital stock does not jump when the economy moves on the R&D line, C does not jump either.

It says to choose consumption in the stochastic regime such that the expected gain from a new technology just matches marginal utility. The trade-off is therefore not between consumption today and capital investment today but between consumption today and the *chance* of having a better technology in the future.

2.2.5. Switch to the deterministic regime

The switch from the stochastic stationary regime to the deterministic regime takes place as soon as a new technology has been developed. All savings will be allocated to capital accumulation, $\theta = 0$, after the development of a new technology as long as the new capital stock is sufficiently small given the new technology. What remains to be found out is how consumption changes.

Replacing the value of the capital stock by utility from current consumption plus marginal utility times savings divided by ρ, the optimality condition for consumption on the R&D line (19) can be expressed alternatively as (cf. appendix)

$$\frac{\rho}{bA^{-\gamma}}u'(C) = u(\tilde{C}) + A^{\gamma+1}u'(\tilde{C})\,[F(\tilde{K},L) - A^{-(\gamma+1)}\tilde{C}] - [u(C) + A^{\gamma}u'(C)\,[F(K,L) - A^{-\gamma}C]], \tag{20}$$

where consumption an instant after a successful R&D project is denoted by $\tilde{C}$. This condition is a rewritten version of the consumption first order condition (19), that must hold in the stochastic regime, where the value functions $V(K)$ and $V(\tilde{K})$ are replaced by expressions given by a maximized Bellman equation (i.e. the Bellman equation (10) with optimal control variables). This condition provides a link between consumption C on the R&D line and consumption $\tilde{C}$ after the development of a new technology. Since consumption changes in a discrete way, this condition will be referred to as consumption jump condition.

This completes the description of the economy and the central planner's solution. When the economy is in the deterministic regime, the evolution of consumption and capital is described by standard laws of motion (15) and (16). When the stock of capital is such that (18) holds with equality, the planner shifts all savings from capital accumulation to R&D and the economy moves from the deterministic to the stochastic regime. In the stochastic regime, the consumption level and the capital stock are constant. As consumption does not jump at this regime switch, both consumption and the capital stock are given by history, i.e. they equal the consumption level and capital stock at the last instant of the deterministic regime. When R&D is successful, a new technology becomes available, the capital stock changes according to (5) and the consumption level jumps as prescribed by (20).

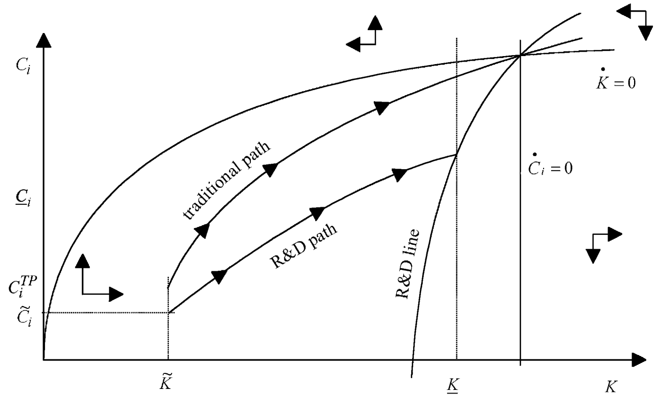

Fig. 1. A phase diagram illustration.

3. An equilibrium with growth and cycles

Is there a path that is characterized by an ever-repeating switch between capital accumulation and R&D? Before we prove the existence of such a path, we present the phase diagram of this economy. This will allow to illustrate the proof and equilibrium properties.

3.1. The phase diagram

The last section has shown that the crucial difference between a standard Ramsey growth model and the model considered here lies in the R&D regime. Illustrating this model therefore simply requires to draw a phase diagram of the deterministic regime and add an 'R&D line', i.e. the loci given by (18) holding with equality, where the economy switches from the deterministic to the stochastic regime.

Such a phase diagram is drawn in Fig. 1. The axes show the stock of capital K and the productivity adjusted level of consumption,

$$C_i = A^{-\gamma} C. \qquad (21)$$

In the deterministic regime, the economy follows (15) and (16) which implies the zero-motion lines

$$\dot{K} = 0 \;\Leftrightarrow\; C_i = F(K,L), \quad \dot{C}_i = 0 \;\Leftrightarrow\; \rho = F_K(K,L). \qquad (22)$$

These zero-motion lines show the advantage for the presentation of results of the assumption that technological progress is restricted to the consumption good sector. If technological progress increased total factor productivity in the investment good and in the R&D sector as well, the zero motion line for consumption would depend on the current vintage γ. Further, the R&D line would also be vintage-dependent. An analysis would then require a variable transformation for capital as well.

Now draw the line on which the planner finds it profitable to invest all savings in R&D. This R&D line follows from (18) and reads with (6), $\theta = 1$ and the productivity adjustment (21),

$$C_i = F(.) - \frac{F_K - \rho}{b[1 - [1 - s]\Omega]}. \tag{23}$$

This line goes through the steady state. For $[1-s]\Omega < 1$, it lies below the zero motion line for capital accumulation to the left of the steady state, where the marginal productivity of capital exceeds the time preference rate, and above it to the right. This reverses when $[1-s]\Omega > 1$. We will assume in what follows that $[1-s]\Omega < 1$ which holds for s sufficiently close or equal to one. [19] This implies that the stock of capital at which the economy stops accumulating capital since the interest rate is too low is smaller than the steady state capital stock in a deterministic Ramsey economy.

The shape of this R&D line can be understood by noting that on this line the planner is indifferent between accumulating an additional unit of capital and financing R&D. First, the R&D line must contain the steady state of the deterministic system. Consumption growth under the deterministic regime is zero when the marginal productivity of capital equals the time preference rate. At the intersection point of the R&D line and the zero motion line for consumption, this indifference requires that the arrival rate be zero since only with a zero arrival rate (expected) consumption growth is zero and equals deterministic consumption growth. Since the arrival rate is zero only if consumption equals total output, the R&D line must go through the steady state.

Second, when R&D is started at a low level of capital, the marginal product of capital is fairly high and a high arrival rate λ is required to obtain this indifference, i.e. equality in (18). Since output is lower as well with a low stock of capital, consumption must fall faster than output to increase an arrival rate that is monotonically increasing in investment, as in (6). This means that the earlier an economy starts with R&D (i.e. the lower the capital stock at the moment R&D starts), the more (more than proportionally) consumption has to be reduced.

Finally, the R&D line is steeper the lower total factor productivity b in R&D. For a given capital stock, a lower b requires more savings and investment into R&D, i.e. lower consumption, in order to make the planner indifferent between capital accumulation and R&D.

[19] This assumption is made for expositional purposes only. It is easy to verify that in the opposite case qualitatively similar results will obtain. In the decentralized economy, this condition reads (use (17) with (26)) $1 - s < b\varpi$.

Since this diagram differs from the Ramsey growth model only in the R&D line, trajectories in this diagram are identical in both models, apart, of course, on and below the R&D line. It is known that the saddle path of the Ramsey growth model can have different forms. A high elasticity of substitution that induces consumers to hardly smooth consumption leads to a quick accumulation of capital and therefore a saddle path that is convex. Inversely, with a low intertemporal elasticity of substitution, the saddle path would be close to the zero motion line for capital accumulation (as drawn) (cf. e.g. Barro and Sala-i-Martin, 1995, Chapter 2.6).

This means that the 'saddle path' of the present model (denoted traditional path[20] in Fig. 1) may intersect the R&D line. Since there exists a traditional path for any initial capital stock $\tilde{K}$, which implies that for K sufficiently low, the traditional path lies between the R&D line and the zero motion line for capital accumulation, a sufficient condition for intersection is that the limit of the slope of the traditional path as it approaches the steady state is lower than the slope of the R&D line in the steady state. This certainly holds when the elasticity of substitution is sufficiently high. In this case, no trajectory exists that leads to the steady state without crossing the R&D line. If it does not intersect, the traditional path is one possible equilibrium path of this economy, i.e. a path that satisfies all optimality and equilibrium conditions.

3.2. *The existence of an equilibrium path with growth and cycles*

Assume the economy starts with some initial capital stock suitable for production with the current vintage, $\tilde{K}$. The economy can jump on the traditional path by choosing the appropriate consumption level C_i^{TP} and approach the long-run steady state if the traditional path lies above the R&D line (as drawn). Since investment into R&D is zero on the traditional path and since the steady state is never reached, the economy will never start investing in R&D.

If the economy starts with a consumption level such as $\tilde{C}_i$, the economy follows standard laws of motion and accumulates physical capital, given an invariant technology vintage γ. Returns to investment in riskless capital accumulation fall such that eventually the planner will find it optimal to use savings for R&D finance. This is when the economy hits the R&D line and savings net of capital depreciation are invested into R&D, $\theta = 1$. The economy will therefore stop growing, consumption will be constant and research for new technologies is undertaken. The aggregate capital stock at this point is denoted by $\underline{K}$ and will be called the R&D capital stock. Eventually, a new

[20] The term saddle path would not be appropriate in the present context since this is not a traditional two-dimensional differential equation system with a unique steady state which can be approached by jumping on the stable arm of the system only.

technology is found and the economy starts with a new initial capital stock $\tilde{K}$. Growth starts again.

This section proves the existence of an equilibrium path with self-replicating cycles and growth, as just informally described, which is the equivalent of this model to a balanced growth path in standard models of growth. An equilibrium path is a feasible path that satisfies all optimality conditions. Let us take as a candidate for such a path the R&D path drawn in Fig. 1. Let the initial and the final capital stock be linked by (5). This path is an equilibrium path if the consumption jump condition (20) implies that the consumption level after development of a new technology is given by $\tilde{C}_i$. If this is the case, starting at $\tilde{C}_i$ implies by the laws of motion (15) and (16) ending at $\underline{C}_i$ and restarting at $\tilde{C}_i$ after successful development of a new technology and so on ad infinitum.

The existence of such a path can be proven under weak parameter restrictions. We now first provide a proof of a theorem and then illustrate the proof of the theorem which shows the very simple idea behind it. The proof requires two assumptions.

Assumption 1 (*Consumption jumps*). Consumption jumps are characterized by a stable, i.e. vintage independent, relationship, where productivity adjusted consumption (21) is smaller after the jump than before,

$$\tilde{C}_i = f(C_{i,.}), \quad \tilde{C}_i < C_i.$$

The assumption of vintage independence is fulfilled under standard specifications of the utility function. To see this, rewrite the consumption jump condition (20), by using productivity adjusted consumption levels (21) and by rearranging, as

$$\frac{\rho}{b} = \frac{u(A^{\gamma}A\tilde{C}_i) - u(A^{\gamma}C_i)}{A^{\gamma}u(A^{\gamma}C_i)} + A\frac{u'(A^{\gamma}A\tilde{C}_i)}{u'(A^{\gamma}C_i)}[F(\tilde{K},L) - \tilde{C}_i] - [F(K,L) - C_i].$$

It is clear that with a utility function of the iso-elastic, $u(C) = \alpha C^{\sigma}$, or logarithmic type, $u(C) = \alpha \ln C$, all terms including the vintage argument γ cancel out and we are left with a link between consumption before and after R&D of $\tilde{C}_i = g_2(C_i)$ which is independent of γ.

We will see in the decentralized economy of Section 4 that the assumption of slumps in productivity adjusted consumption holds for weak parameter restrictions.

Assumption 2 (*Consumption jump from steady state*). If a consumption jump took place starting from the steady state, consumption after the jump would lie above the traditional path,

$$f(C_i^{\text{st}}) > C_i^{\text{tp}}.$$

We can now prove

Theorem 2 (Existence of an equilibrium path with growth and cycles). *Under Assumptions 1 and 2, there exists an equilibrium path with self-replicating cycles as the R&D path drawn in Fig. 1.*

Proof. Choose some capital stock $\underline{K}$ on the R&D line as a candidate member of the equilibrium path. This determines the consumption level $\underline{C}_i$ on the R&D line. By the destructive effect (5) and by the consumption jump condition, this fixes $(\tilde{K}, \tilde{C}_i)$. If $\underline{K}$ was chosen too small, consumption $\tilde{C}_i$ will, by Assumption 1, be negative. Increasing $\underline{K}$ will eventually imply $\tilde{C}_i = 0$. The economy will then end up, after having followed deterministic laws of motion (15) and (16), on the R&D line with a consumption level of zero, as all output has always been invested. This shows that the capital stock $\underline{K}$, which is strictly positive, was still chosen too small.

Now start with a $\underline{K}$ close to the steady state value. By Assumption 2, this implies a consumption level $\tilde{C}_i$ above the traditional path. The capital level $\underline{K}$ was obviously chosen too large. Decreasing the capital level $\underline{K}$ will eventually imply that the economy finds itself on the traditional path after the jump which shows that the capital stock was still too large.

Denote the mapping implied by differential equations of the deterministic regime by $g : (\tilde{K}, \tilde{C}_i) \to (\underline{K}^*, \underline{C}_i^*)$. Note that g is continuous. The last two arguments have shown that there is an upper $\underline{K}$ for which $\underline{C}_i^* > \underline{C}_i$ and a lower $\underline{K}$ for which $\underline{C}_i^* < \underline{C}_i$. As g is continuous, there must be a $\underline{K}^*$ where $\underline{C}_i^* = \underline{C}_i$. □

The theorem and its proof can be nicely illustrated with the help of Fig. 1. Assumption 2 says that the consumption jump condition implies that jumping from the steady state back (or from a value very close to the steady state as in the steady state, investment is zero and therefore new technologies cannot be invented) to $\tilde{K}$ implies that the resulting consumption level $\tilde{C}_i$ lies above the consumption level C_i^{TP} on the saddle path. Choosing an initial consumption level close to C_i^{TP} therefore cannot put the economy on an equilibrium path with self-replicating cycles as after development of a new technology the economy finds itself above the saddle path and will never again invest in R&D.

Likewise, by Assumption 1, if the economy starts with a consumption level that is very low (close to zero or zero), it will end up with a negative consumption level after one innovation. As there is a monotonically increasing continuous functional relationship between the initial consumption level $\tilde{C}$ and the consumption level after the first innovation, there is a consumption level $\tilde{C}^*$ for which the economy finds itself on an equilibrium path with positive expected long-run growth rates and short-run fluctuations.

3.3. Equilibrium properties

On an equilibrium R&D path, the initial capital stock is constant across vintages. The initial consumption level, however, rises from vintage to vintage by A. This follows directly from the link between consumption C and productivity adjusted consumption C_i in (21) and the fact that the initial productivity adjusted consumption level $\tilde{C}_i$ is the same for all vintages. The same is true for the consumption and capital stocks at the end of an R&D path. No result is available for consumption before and after the development of a new technology. The productivity adjusted level (21) as shown in the phase diagram falls but the consumption level C may still increase as γ increases. In the deterministic regime, consumption and capital smoothly increase but remain constant in the stochastic regime. Similar paths hold for the marginal productivity of labour (rising) and capital (falling).

The capital stock at the beginning of a cycle must be smaller than at the end of a cycle. Hence, as technological progress is restricted here to the consumption good sector, the destructive part is a necessary condition for cycles to obtain. If the introduction of new technologies did not cause obsolescence of old production units, the model would be similar to Aghion and Howitt (1992) extended for an explicit study of household behaviour (as in Wälde (1999a), who does not study capital accumulation or cycles, however). Imagine that parameters are such that (7) implies $\tilde{K} = K$. Then the economy would permanently stay on the R&D line, once it is reached, and total output and consumption would grow whenever a new technology is developed.

Investment into R&D follows a cyclical process. If research was not only required for the development of new technologies but also for how to best adapt technologies for a particular industry or firm, R&D expenditure would be observed all over the time and not only at specific points of the cycle.

If the traditional path does not intersect the R&D line as drawn in Fig. 1, the economy is characterized by multiple equilibria. In addition to the equilibrium path with growth and cycles whose existence was just proven, the traditional saddle path is an equilibrium path as well. Apart from a singular case, welfare of these two paths differ. It can be conjectured that for reasonable parameter values the path with growth and cycles Pareto dominates the traditional path that leads to a steady state.[21] We therefore assume that the planner would indeed choose the path with long-run growth and cycles.

[21] Reasonable parameter values means e.g. that A should be sufficiently larger than unity. Should A equal unity, there would be no reason to invest into R&D and approaching the steady state would yield higher utility. Similar arguments can be made for other parameters like the productivity parameter b for R&D. It would be interesting to find out numerically where these thresholds lie.

4. Decentralization

This section presents a decentralized version of the above model. All equilibrium properties will continue to hold, notably the zero–one decision for R&D. This preserves the tractability of the model also for the decentralized version. In addition, the consumption jump condition will be much simpler. The underlying economic behaviour suggests, however, that the decentralized equilibrium is not socially optimal.

Assume a large number of households whose preferences are identical to those of the planner in (8). Given the technologies as described above, it can be shown along the lines of Wälde (1999b) that a household's budget constraint reads

$$\mathrm{d}a = (ra + w - i - e)\mathrm{d}t + \left(p_I \varpi \frac{i}{J} - sa\right)\mathrm{d}q. \tag{24}$$

The wealth of households is denoted by a on which interests r are paid. Wage income is given by w while i and e are R&D and consumption expenditure, respectively. The price of the investment good is denoted by p_I and J is aggregate investment into R&D; the rest of the notation is as above.

One of the commonplaces of economics is that R&D in a decentralized economy requires some form of imperfect competition as this allows those investing in R&D to recover their investment in R&D. This budget constraint shows that R&D and perfect competition can be reconciled if the outcome of R&D is not only a better technology but also some tangible good, ϖ in this case. Factor payments to those that own the new production unit ϖ cover R&D payments made before, even under perfect competition.

The household's solution of it's maximization problem yields the same dichotomy as above, provided that households can be described by the concept of a representative agent and that the R&D sector, i.e., the arrival rate in (6), is characterized by constant returns to scale, as is common for perfectly competitive economies. Investment therefore follows (a proof and further discussion can be found in Wälde, 1999b)

$$i = \left\{ \begin{array}{c} 0 \\ ra + w - e \end{array} \right\} \Leftrightarrow r - \rho - \lambda[1 - [1 - s]\Omega] \left\{ \begin{array}{c} > \\ = \end{array} \right\} 0.$$

An aggregation of these optimality conditions leads to exactly the same equations of motion as in the central planner's economy. In the deterministic regime, aggregate consumption follows (15), the capital stock follows (16) and both variables are constant in the stochastic regime. When the stochastic regime ends, i.e. after a new technology has been found, the new aggregate capital stock is (and of course has to be) given by (5). The only but very interesting difference between the centralized and decentralized equilibrium lies in the jump of consumption after the discovery of a new technology. To see this, following Wälde (1999b), consider a household's Bellman equation

$$\rho V(a) = \max_{e,i}\{u(c) + V'(a)[ra + w - i - e] + \lambda[V(\tilde{a}) - V(a)]\}.$$

In analogy to above, a tilde ($\sim$) indicates the value of a variable immediately after the development of a new technology, i.e. from (24) $\tilde{a} = iJ^{-1}p_I\varpi + (1-s)a$. Computing the derivative with respect to R&D investment i gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}i}&\{u(c) + V'(a)[ra + w - i - e] + \lambda[V(\tilde{a}) - V(a)]\}\\ &= -V'(a) + \lambda V'(\tilde{a})\frac{p_I\varpi}{J}\\ &= -V'(a) + b\varpi V'(\tilde{a}).\end{aligned}$$

where we used (6) and $J = p_I R_{\mathrm{R\&D}}$. This should be compared with the planner's derivative (12). The planner compares the marginal gain from an additional unit of capital under the present technology with the *marginal expected* gain from a new technology, taking the change in the capital stock through innovation into account. Households trade off the marginal gain from an additional unit of capital with the *expected marginal* gain resulting from a new technology. Households take the arrival rate as given and consider the variation in their wealth after successful R&D; the planner takes the aggregate capital stocks before and after innovation as given and considers the variation in the arrival rate. Households can influence their individual stock of wealth after R&D by choosing their individual R&D investment level while the planner cannot as the aggregate capital stock is given at each point in time. As households neglect the effect of their investment on the arrival rate and on the wealth of other households, the overall investment level into R&D tends to be too low.

The crucial difference from a modelling perspective is that the decentralized first order condition contains only derivatives of the value function while the centralized first order condition contains the value function itself. This has the convenient implication that the consumption jump condition in a decentralized economy takes a much simpler form than the consumption jump condition (20) in the centralized economy. Inserting the consumption first order condition (14) gives

$$u'(c)\frac{1}{p} = b\varpi u'(\tilde{c})\frac{1}{\tilde{p}}. \tag{25}$$

From technologies (1), the consumption good prices are given by $p = A^{-\gamma}p_I$ and $\tilde{p} = A^{-(\gamma+1)}p_I$ and this reads (apply $(u')^{-1}$ to the individual jump condition and sum over all individuals)

$$u'(C) = Ab\varpi u'(\tilde{C}). \tag{26}$$

Note that this condition does not depend on the degree of obsolescence s of old capital. This parameter does play a role in the aggregate evolution of the economy, however, and affects household decisions indirectly through prices. Further, if households do not obtain any new capital from successful R&D, i.e. if $\varpi = 0$, no investment in R&D would take place. This confirms the argument made above that under perfect competition there can be investment into R&D provided that R&D leads not only to a disembodied new technology but also to a private tangible good that embodies this new technology.

From a qualitative perspective, the aggregate economy behaves as was illustrated for the centrally planned economy in Fig. 1. Letting the economy start with an initial capital stock $\tilde{K}$ and assuming for a moment an initial consumption level $\tilde{C}_i$, the economy follows the R&D path and eventually hits, in finite time, the R&D line. The amount of resources invested in R&D is then the same as above which implies that the expected duration of the stochastic regime is identical. Once the technology is found, the jump in consumption now differs. Comparing the size of the jump (does consumption jump too much or too little compared to the social optimum?) between the centralized (20) and decentralized (26) economy would be very interesting but is too complex to be done in this short section. It is therefore left for future work.[22]

The proof for the existence of a cyclical equilibrium follows closely the proof given above. Assumption 1 now requires from (26) with a CES utility function $u(C) = C^{\sigma}$ that

$$A^{-1}(Ab\varpi)^{1/(1-\sigma)} < 1 \tag{27}$$

and, under Assumption 2, an almost identical version of Theorem 2 can be proven.[23]

A further interesting difference between the centralized and decentralized economy consists in the choice of the equilibrium path (if parameter values are such that two paths exist, if e.g. the elasticity of intertemporal substitution is sufficiently low). While a planner will choose the path that yields higher welfare, the decentralized economy is characterized by a coordination problem. If all individuals choose their consumption level such that the economy finds itself on the traditional path, there will be no long-run growth and no cycles. If beliefs are such that the economy settles on the R&D path,

[22] As the size of jumps are bound to differ, equilibrium paths will generally not be the same. Hence, despite qualitative resemblance, there will be quantitative differences.

[23] Unlike for Assumption 1, no explicit condition for Assumption 2 can be provided in the decentralized equilibrium as no closed-form solution of the saddle path is known. Numerical examples have shown, however, that Assumption 2 does not impose strong restrictions on the equilibrium parameter space.

the economy will grow in the long-run, though with ever fluctuating growth rates.[24]

5. Conclusion

Creative destruction causes an economy to permanently diverge from its balanced growth path. Each new technology pushes the economy on a higher productivity level but at the same time renders some old production units obsolete. A higher productivity level as well as the destruction of old production units implies higher returns to capital accumulation. New technologies therefore lead to faster capital accumulation. When the capital stock has reached a sufficiently high level and returns have therefore fallen, capital accumulation is no longer profitable compared to the development of new technologies. Savings will shift to financing R&D. Once a new technology becomes available, the effects of creative destruction become visible again and the next cycle starts.

These results were derived in a conceptually very simple extension of the textbook Ramsey continuous-time growth model. From a modelling perspective, all that is required to jointly study long-run growth and short-run business fluctuations is to add an R&D line into an otherwise standard phase-diagram. As such an extension is simple once the basic mechanisms are understood, this approach should prove useful for understanding the link between cycles and long-run growth in other models as well.

Equilibrium properties are qualitatively identical for the centralized and decentralized economy. The decentralized economy is not optimal, however, as individuals neglect the positive externality of their investment in a new technology on the economy as a whole. Despite perfect competition in all sectors, individuals do invest in R&D as the R&D process does not only yield a new technology but also a tangible private good and new machine. As it will be owned by private investors, its output (its value marginal product) will be used to cover R&D costs.[25]

An alternative channel that creates business cycles as well, without reliance on obsolescence of old production units, would become visible if the

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[24] There are various approaches in the literature to this focal point problem. Baron and Kalai (1993) study equilibrium selection in a game theoretical setup while Moore (1993) shows how learning can help selecting an equilibrium. These issues go far beyond the scope of this present paper.

[25] As one referee pointed out, there is an interesting link to the Coase theorem as it was recently amended by Dixit and Olson (2000): When bundling a collective good (the new technology) with a private good (the new machine), the collective good will be provided.

investment technology was subject to technological change as well.[26] A new technology would then have a double creative aspect in that it increases total factor productivity not only in the consumption good but also in the investment good sector. Growth rates would then fluctuate since the increase of total factor productivity in the investment good sector increases the steady state capital stock. In the current formulation, the steady state capital stock is independent of the current vintage as shown by (22). As new technologies would make accumulation of capital relatively more profitable (the steady state of the phase diagram would have moved to the right), the development of a new technology would be followed by accumulation of additional capital. At a certain point, the economy would again be sufficiently close to the new steady state and research for new technologies would again be undertaken.

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Appendix A

This appendix proves the Theorem in (18). The proof is in 3 parts. Part I proves a proposition, Part II proves 3 lemmas and Part III uses these results to prove the Theorem.

A.1. Part I

Proposition.

$$\left.\begin{array}{l}\theta=0\\ \theta\in[0,1]\\ \theta=1\end{array}\right\}\Leftrightarrow F_K(.)\left\{\begin{array}{c}>\\=\\<\end{array}\right\}\rho+\lambda(.)\left[1-[1-s]\frac{V'(\tilde{K})}{V'(K)}\right].$$

[26] As noted before, this was analysed in an earlier version of this paper (Wälde, 1998, Chapter 4).

Proof. The proof is in 3 steps.

Step 1: $\theta \in [0,1] \Leftrightarrow F_K(.) = \rho + \lambda(.)\,[1 - [1-s]\frac{V'(\tilde{K})}{V'(K)}]$.

Since by (13), an agent is indifferent, i.e. $\theta \in [0,1]$, iff $b[V(\tilde{K}) - V(K)] = V'(K)$, we first prove

Lemma 1.

$$b[V(\tilde{K}) - V(K)] = V'(K) \Rightarrow F_K(.) = \rho + \lambda(.)\left[1 - [1-s]\frac{V'(\tilde{K})}{V'(K)}\right].$$

Proof. When the planner is indifferent, she switches from $\theta = 0$ to $\theta = 1$. Hence, indifference implies $\theta = 1$. The partial derivative of the maximized Bellman equation (10) with respect to K reads after inserting the consumption first order condition and $b[V(\tilde{K}) - V(K)] = V'(K)$, which holds under indifference, and after some rearranging (cf. Appendix A)

$$[\rho + \lambda(.) - F_K(.)]V'(K) - \lambda(.)V'(\tilde{K})\,[1-s] = V''(K)\,[1-\theta][F(.) - A^{-\gamma}C]. \tag{28}$$

$\theta = 1$ then implies this lemma. □

Lemma 2.

$$F_K(.) = \rho + \lambda(.)\left[1 - [1-s]\frac{V'(\tilde{K})}{V'(K)}\right] \Rightarrow b[V(\tilde{K}) - V(K)] = V'(K).$$

Proof. The partial derivative of the maximized Bellman equation with respect to K in all generality is given by (cf. Appendix A)

$$\begin{aligned}
&[\rho + \lambda(.)]V'(K) - \lambda(.)V'(\tilde{K})\,[1-s] \\
&\quad = V''(K)\,[1-\theta]\,[F(.) - A^{-\gamma}C] \\
&\qquad - [V'(K) - \lambda'(.)\,[V(\tilde{K}) - V(K)]]\theta'(K)\,[F(.) - A^{-\gamma}C] \\
&\qquad + [[1-\theta]V'(K) + \theta\lambda'(.)\,[V(\tilde{K}) - V(K)]]F_K(.).
\end{aligned} \tag{29}$$

When $F_K(.) = \rho + \lambda(.)\,[1 - [1-s]V'(\tilde{K})/V'(K)] \Leftrightarrow F_K(.)V'(K) = [\rho + \lambda(.)]V'(K) - \lambda(.)\,[1-s]V'(\tilde{K})$, this derivative reads

$$\begin{aligned}
F_K(.)V'(K) = {} & V''(K)\,[1-\theta]\,[F(.) - A^{-\gamma}C] \\
& - [V'(K) - \lambda'(.)\,[V(\tilde{K}) - V(K)]]\theta'(K)\,[F(.) - A^{-\gamma}C] \\
& + [[1-\theta]V'(K) + \theta\lambda'(.)\,[V(\tilde{K}) - V(K)]]F_K(.)
\end{aligned}$$

$$\begin{aligned}\Leftrightarrow\ \ 0 &= V''(K)[1-\theta][F(.)-A^{-\gamma}C] \\ &\quad -[V'(K)-\lambda'(.)[V(\tilde{K})-V(K)]]\theta'(K)[F(.)-A^{-\gamma}C] \\ &\quad -[V'(K)-\lambda'(.)[V(\tilde{K})-V(K)]]\theta F_K(.) \\ \Leftrightarrow &[V'(K)-\lambda'(.)[V(\tilde{K})-V(K)]][\theta'(K)[F(.)-A^{-\gamma}C]+\theta F_K(.)] \\ &= V''(K)[1-\theta][F(.)-A^{-\gamma}C].\end{aligned}$$

Assume that $\theta < 1$. Then the right-hand side is negative, and $V'(K) - \lambda'(.)[V(\tilde{K}) - V(K)] < 0$ must hold as $\theta'(K)[F(.) - A^{-\gamma}C] + \theta F_K(.) > 0$. But $V'(K) - \lambda'(.)[V(\tilde{K}) - V(K)] < 0$ implies $\theta = 1$, which is a contradiction. Hence, θ must equal unity. Then, the right-hand side is zero which implies $V'(K) - \lambda'(.)[V(\tilde{K}) - V(K)] = 0$. □

Keeping (13) in mind, this completes step 1.

Step 2: $\theta = 1 \Leftrightarrow F_K(.) < \rho + \lambda(.)\left[1 - [1-s]\frac{V'(\tilde{K})}{V'(K)}\right]$.

Lemma 3.

$$F_K(.) < \rho + \lambda(.)\left[1 - [1-s]\frac{V'(\tilde{K})}{V'(K)}\right] \Rightarrow \theta = 1.$$

Proof. Assume an interior solution with $0 < \theta < 1$ holds. Then, (28) must hold and inserting $F_K(.) < \rho + \lambda(.)[1 - [1-s]V'(\tilde{K})/V'(K)]$ requires that the right-hand side of (28) is positive. As $V''(K) < 0$, this could be only fulfilled if $1 - \theta < 0$ or $F(.) - A^{-\gamma}C < 0$. As the former is technologically unfeasible and the latter is excluded, as we focus on time paths where savings are positive, an interior solution cannot exist.

Assume $\theta = 0$. Then, the derivative of the Bellman equation reads $[\rho - F_K(.)]V'(K) = V''(K)[F(.) - A^{-\gamma}C]$. As with $F_K(.) < \rho$, the left-hand side is positive, $\theta = 0$ cannot be a solution either. Hence, θ must be equal to unity. □

Lemma 4.

$$\theta = 1 \Rightarrow \rho + \lambda(.)\left[1 - [1-s]\frac{V'(\tilde{K})}{V'(K)}\right] > F_K(.).$$

Proof. When $\theta = 1$, the maximized Bellman equation reads

$$\rho V(K) = u(C) + \lambda(F(.) - A^{-\gamma}C)[V(\tilde{K}) - V(K)]. \tag{30}$$

Computing the derivative of this equation and inserting the consumption first order condition gives (cf. Appendix A)

$$[\rho + \lambda(.)]V'(K) - \lambda(.)\,[1-s]V'(\tilde{K}) = \lambda'(.)F_K[V(\tilde{K}) - V(K)]. \qquad (31)$$

From (13), $\theta = 1$ implies $b[V(\tilde{K}) - V(K)] > V'(K)$, hence

$$[\rho + \lambda(.)] - \lambda(.)\,[1-s]\frac{V'(\tilde{K})}{V'(K)} = F_K \frac{\lambda'(.)\,[V(\tilde{K}) - V(K)]}{V'(K)} > F_K. \qquad \square$$

Step 3: $\theta = 0 \Leftrightarrow F_K > \rho + \lambda\left[1 - [1-s]\frac{V'(\tilde{K})}{V'(K)}\right]$.

The proof of this step directly follows from Steps 1 and 2 since they are mutually exclusive.

A.2. Part II

Lemma 5.

$$\theta = 0 \;\Rightarrow\; F_K > \rho + \lambda[1 - [1-s]\Omega]$$

Proof. From the above proposition we know

$$\theta = 0 \;\Rightarrow\; F_K < \rho + \lambda\left[1 - [1-s]\frac{V'(\tilde{K})}{V'(K)}\right].$$

Since (14) holds both when $\theta = 0$, i.e. in the deterministic regime, and when the planner is indifferent, $V'(\tilde{K}) = u'(\tilde{C})A^{\gamma+1}$ and $V'(K) = u'(C)A^{\gamma}$. With the definition in (17), we can replace according to $\Omega = V'(\tilde{K})/V'(K)$. $\square$

Lemma 6.

$$\theta \in [0,1] \;\Rightarrow\; F_K = \rho + \lambda[1 - [1-s]\Omega].$$

Proof. From the above proposition we know $\theta \in [0,1] \Rightarrow F_K = \rho + \lambda[1 - [1-s]V'(\tilde{K})/V'(K)]$. We can replace here again $\Omega = V'(\tilde{K})/V'(K)$. $\square$

Lemma 7. When the economy starts with a capital stock that implies $\theta = 0$, *the economy can be in two states only*, $\theta = 0$ *or* $\theta \in [0,1]$, *where* $\theta = 1$.

Proof. When $\theta = 0$, the economy accumulates capital. This might continue forever or at some point a capital stock is reached where the planner becomes indifferent between capital accumulation and financing R&D. If she is indifferent, she switches from $\theta = 0$ to $\theta = 1$. As $\theta = 1$ implies that the capital stock is constant, the economy will remain in this state of indifference (until a new technology is found). $\square$

A.3. Part III

By employing results obtained so far and the assumption that the economy starts with a capital stock that implies $\theta = 0$, we can now prove the theorem.

Proof. As the economy starts with $\theta = 0$ and by Lemma 7, the optimal share is either zero or one. As $(x \Rightarrow y) \Rightarrow (\neg y \Rightarrow \neg x)$, Lemmas 5 and 6 complete the proof. □

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