An example for a closed-form solution for a system of linear partial differential equations

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We analyse a system consisting of two linear partial differential equations. We present a closed-form solution which has a "gamma-structure", i.e. it is of form xe^{-x}.

Keywords: linear partial differential equation system, analytical solution

1 Introduction

Consider a function $p(a,z,t) : \mathbb{R} \times \{w,b\} \times \mathbb{R}_0^+ \to \mathbb{R}$. We require that this function satisfies the following system of partial differential equations (PDEs),

$$\text{PDE}^w \equiv \frac{\partial}{\partial t} p\left(a,w,t\right) - m_w \frac{\partial}{\partial a} p\left(a,w,t\right) + sp\left(a,w,t\right) - \mu p\left(a,b,t\right) = 0, \quad \text{(1a)}$$

$$\text{PDE}^b \equiv \frac{\partial}{\partial t} p\left(a,b,t\right) - m_b \frac{\partial}{\partial a} p\left(a,b,t\right) + \mu p\left(a,b,t\right) - sp\left(a,w,t\right) = 0, \quad \text{(1b)}$$

where s, μ, m_w and m_b are known real constants. Further we assume that $m_w \neq m_b$.

Those equations are motivated in economic papers by Bayer and Wälde (2010 a,b,c). The authors analyse the distribution of wealth a in a world of uncertain labour income, i.e. labour income z that moves stochastically between a low state b and a high state w. This stochastic movement is modeled using two mutually independent Poisson processes. Arrival rates are s for the transition from state w to b and μ for the opposite direction. The state w can be called "employment" and state b "unemployment". Individuals can save in this setup implying a distribution of wealth that is a function of labour market history. The above linear system results from Fokker-Planck equations describing the joint density of a and z under the assumption of a utility function characterised by so-called "constant absolute risk aversion".

As far as we can tell, general closed form solutions for a linear PDE system (1) do not seem to exist.[2] A posting by Serre (2011) supports this opinion by stating that a general closed-form solution for a linear PDE system does not exist. Special solutions, i.e. solutions for special initial functions, are well-known. An example are exponential-type solutions of the form $e^x e^t$ (Israel, 2011). After having consulted, inter alia, Polyanin (2001, 2011), Polyanin et al. (2001), Evans (2010), Farlow (1993) and Zachmanoglou and Thoe (1986), we concluded that solutions of the "gamma-type" suggested here are not as widely known as one would expect.[3]

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[2]In parallel work, ten Thije Boonkkamp and ourselves solve this system using the method of characteristics as presented e.g. in Mattheji, Rienstra and ten Thije Boonkkamp (2005). We obtain a system of integral equations or ordinary differential equations on characteristic lines that do not suggest an obvious closed-form solution.

[3]We would call a solution widely known if it appears in textbooks. We would be very happy to receive hints to the journal literature if closed-form solutions are available there.

We suggest and prove that there are closed-form solutions of the type

$$p(a,w,t) = (c_{0w} + c_{aw}a + c_{tw}t)\, e^{-c_1 t - c_2 a}, \tag{2a}$$
$$p(a,b,t) = (c_{0b} + c_{ab}a + c_{tb}t)\, e^{-c_1 t - c_2 a}. \tag{2b}$$

where c_{0w}, c_{0b}, c_{aw}, c_{ab}, c_{tw}, c_{tb}, c_1 and c_2 are parameters which are to be determined. These solutions could be called gamma-type solutions, given that their structure reminds of the gamma-distribution.

The remaining parts of this paper is structured as follows. Chapter 2 gives a theorem showing that for certain parameter restrictions, our guess in (2) is indeed a solution to (1). We also give a short sketch of the proof - which can be found in full detail in the appendix. Chapter 3 presents a numerical illustration of our solution.

2 A "gamma-type" solution

We try to keep this chapter as short as possible and at the same time include the key steps of the proof of our main theorem. Interested readers can find the complete proof in the appendix.

2.1 Main result

Proposition 1 *Consider the function*

$$p(a,z,t) = (c_{0z} + c_{az}a + c_{tz}t)\, e^{-c_1 t - c_2 a}, \qquad z \in \{w, b\} \tag{3}$$

where $c_{0w}, c_{aw}, c_{tw}, c_{0b}, c_{ab}, c_{tb}, c_{1,}c_2 \in \mathbb{R}\backslash\{0\}$ *are constants and define*

$$C_b \equiv -c_1 + c_2 m_b + \mu, \tag{4a}$$
$$C_w \equiv -c_1 + c_2 m_w + s. \tag{4b}$$

Then, $p(a,w,t)$, $p(a,b,t)$ *are solutions to the system of PDEs given in (1) if and only if all of the following five conditions hold*

$$\frac{C_b}{s} = \frac{\mu}{C_w} \tag{5a}$$
$$\frac{c_{tw}}{c_{tb}} = \frac{\mu}{C_w}, \tag{5b}$$
$$\frac{c_{aw}}{c_{ab}} = \frac{\mu}{C_w}, \tag{5c}$$
$$\left(s m_w \mu + C_w^2 m_b\right) c_{aw} = \left(C_w^2 + s\mu\right) c_{tw}, \tag{5d}$$
$$c_{tw} - c_{aw} m_w = -c_{0w} C_w + c_{0b}\mu, \tag{5e}$$

where μ, s, m_b *and* m_w *are given parameters as described after (1).*

Proof. (sketch - see appendix for complete version) The proof is completely constructive. We start by substituting our (3) and its derivatives into the PDE system (1). This yields

$$0 = c_{tw} - c_{aw} m_w + c_{0w} C_w - c_{0b}\mu + (c_{tw} C_w - c_{tb}\mu)\, t + (c_{aw} C_w - c_{ab}\mu)\, a,$$
$$0 = c_{tb} - c_{ab} m_b + c_{0b} C_b - c_{0w} s + (c_{tb} C_b - c_{tw} s)\, t + (c_{ab} C_b - c_{aw} s)\, a.$$

Taking into consideration, that $a \in \mathbb{R}$ and $t \in \mathbb{R}_0^+$, these equations can be satisfied only if the terms in front of a and t are zero. Imposing this on parameters, we also need the remaining terms to equal zero. Imposing this as well, we end up with a system of six equations to determine the eight parameters $c_{0w}, c_{aw}, c_{tw}, c_{0b}, c_{ab}, c_{tb}, c_{1,}c_2$.

Analysing those equations, we find that one of these six equations is redundant and we end up with the five conditions in (5). We then choose c_2, c_{tw}, c_{0b} as free parameters. Once we have chosen c_2, we can determine c_1 by solving the quadratic equation (5a). As we can choose c_2, we always find at least one real solution for c_1. Subsequently, fixing c_{tw} and the previous results, we use (5b) to compute c_{tb} and (5d) for c_{aw}. The next step is given by (5c) an the computation of c_{ab}. Finally, given all previous results and fixing c_{0b}, we can use (5e) to determine the missing c_{0w}. ■

2.2 A first example

For a first application, we set $m_w = s = -1$ and $m_b = \mu = 1$, i.e. we want to solve

$$\frac{\partial}{\partial t}p\left(a,w,t\right)+\frac{\partial}{\partial a}p\left(a,w,t\right)-p\left(a,w,t\right)-p\left(a,b,t\right)=0, \tag{6a}$$

$$\frac{\partial}{\partial t}p\left(a,b,t\right)-\frac{\partial}{\partial a}p\left(a,b,t\right)+p\left(a,b,t\right)+p\left(a,w,t\right)=0. \tag{6b}$$

One solution is

$$p(a,w,t)=te^{2a},\ p(a,b,t)=\left(1+t\right)e^{2a}, \tag{7}$$

i.e. with the notation of theorem 1: $c_1 = c_{0w} = c_{ab} = c_{aw} = 0$, $c_{0b} = c_{tb} = c_{tw} = 1$ and $c_2 = -2$.[4]

To verify that, we compute partial derivatives,

$$\frac{\partial}{\partial t}p\left(a,w,t\right)=e^{2a},\ \frac{\partial}{\partial a}p\left(a,w,t\right)=2te^{2a},$$
$$\frac{\partial}{\partial t}p\left(a,b,t\right)=e^{2a},\ \frac{\partial}{\partial a}p\left(a,b,t\right)=2(1+t)e^{2a},$$

substitute them plus our guess (7) into the initial system (6) and see that (6) holds.

3 Numerical illustration

We now provide numerical examples for solutions. This is of importance for applications as it illustrates which values the parameters can take.[5] The first subsection provides two equations that determine m_w and m_b. This is an outcome of the economic system in the background. Someone interested in the numerical illustration per se can skip this section and take values for m_w and m_b as given.

Parameters for our numerical illustrations are given by

$$r=0.0198,\ s=0.014,\ \rho=0.02,\ w=2,\ b=1,\ \mu=0.079,\ \gamma=0.2. \tag{8}$$

While the values can be motivated from the economic model as well, they are relatively arbitrary at this point.

[4] To get this solution, we chose $c_2 = -2$ and $c_{tw} = c_{0b} = 1$. The value of c_2 together with the values of m_w, m_b, μ and s imply that we only have one solution for c_1, i.e. $c_1 = 0$.

[5] All codes are available at www.waelde.com/pub right next to this paper.

3.1 Values of m_w and m_b

The optimal consumption path under a utility function with constant absolute risk aversion is given by $c(a, z) = ra + z + m_z,\ \ z \in \{b, w\}$, provided that

$$r\left[1 + m_w\gamma\right] - s - \rho + se^{\gamma[w-b+m_w-m_b]} = 0,$$
$$r\left[1 + m_b\gamma\right] - \mu - \rho + \mu e^{-\gamma[w-b+m_w-m_b]} = 0,$$

hold. Here, r is the constant interest rate, γ is the CARA-parameter, ρ is the discount factor and s and μ are the arrival rates of the Poisson processes. These two equations result from the Bellman equations of the maximization problems (one for each state). Using our parameter values from (8), we get

$$m_w = -0.0762 < 0,\ \ m_b = 0.7470 > 0.$$

3.2 Determination of $p(a, z, t)$

Having three parameters we can choose to our liking, we split up this section into three parts. In each part, we fix two out of the three free parameters and the third parameter is allowed to vary within an interval. Using those values we compute the parameters fixed by our conditions (5) and analyse their behaviour.

As the proof of theorem 1 has shown, we can have up to two real values for c_1 given a fixed value of c_2. We denote these two solutions by c_{1+} and c_{1-} here.

3.2.1 Fix c_2 and c_{tw}

For our fist illustration we fix $c_2 = 1$ and $c_{tw} = 2$. We let the remaining free parameter c_{0b} vary between -2 and 2, as plotted on the horizontal axis of the next figure.

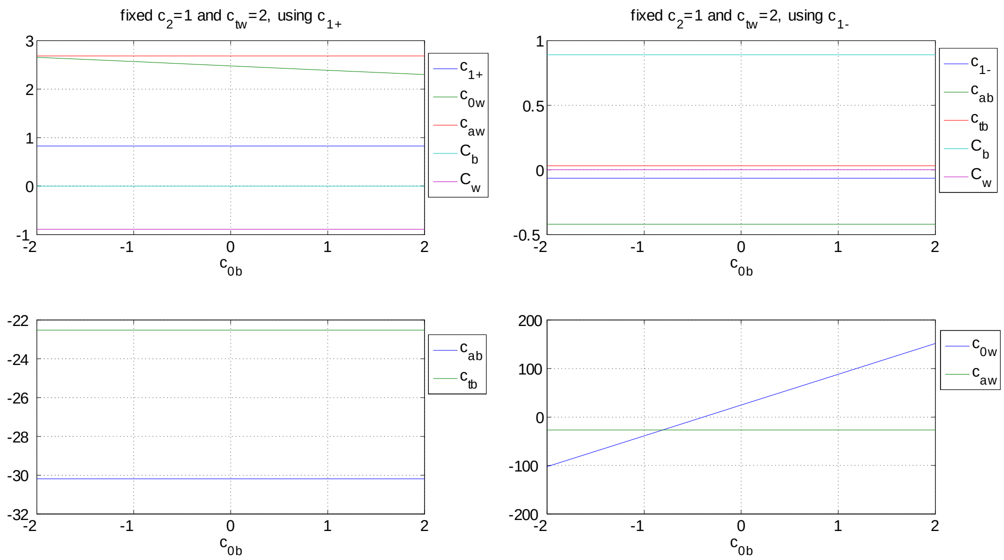

Figure 1 *Parameter values for $c_2 = 1$, $c_{tw} = 2$ and $-2 \leq c_{0b} \leq 2$, using c_{1+} in the left panel and c_{1-} in the right panel.*

The parameter values for c_{1+} are illustrated in the left panels, those for c_{1-} in the right panels. In the case of c_{1+}, we see that all parameters are independent of c_{0b}, apart from c_{0w}. Looking at the conditions in (5) immediately shows that this obviously needs to hold true. As the condition do not distinguish between different values of $c_1$we observe the same behaviour for c_{1-}, but with obviously different absolute values.

3.2.2 Fix c_{tw} and c_{0b}

Let $c_{tw} = 1$ and $c_{0b} = 2$.

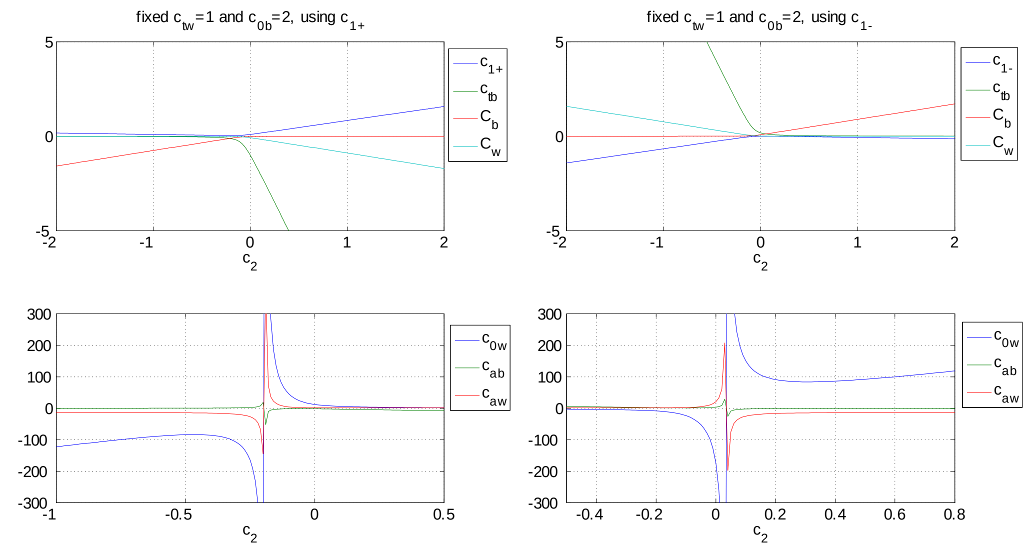

Figure 2 *Parameter values for $c_{tw} = 1$, $c_{0b} = 2$ and different values of c_2, using c_{1+} in the left panel and c_{1-} in the right panel.*

Here, as c_2 is not fixed, we have no constant parameters. Instead there exists a pole for c_{0b},c_{ab} and c_{aw} as shown in the second row in the figure. The value of that pole depends on C_w.[6] c_1+ is a flat, opened up and positive valued parabola, whereas c_1- is also flat but open down with both positive and negative values. The left and right panel seems to be connected by a symmetry issue, which is not further analysed here.

3.2.3 Fix c_2 and c_{0b}

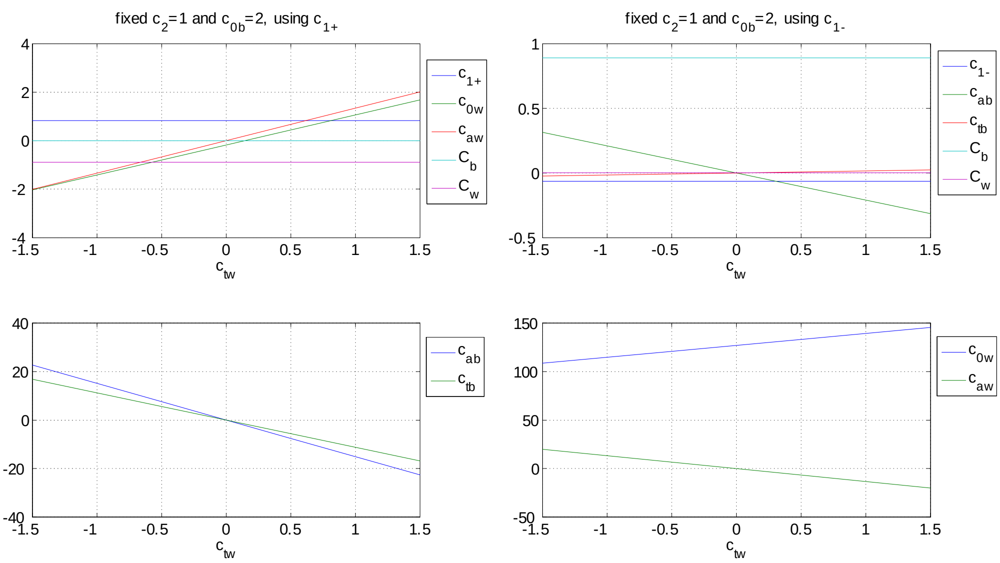

Figure 3 *Parameter values for $c_2 = 1$, $c_{0b} = 2$ and $-1.5 \leq c_{tw} \leq 1.5$, using c_{1+} in the left panel and c_{1-} in the right panel.*

[6] Looking e.g. at c_{aw} in (5d) the denominator equals 0 if $C_w = \sqrt{-\left(sm_w\mu\right)/m_b}$.

For both, c_{1+} in the left panel and c_{1-} in the right panel, parameters c_{0w}, c_{aw}, c_{ab} and c_{tb} depend on c_{tw}. The remaining three parameters are constant. Obviously the absolute values in both cases are different.

3.2.4 Looking at the shape of $p(a, z, t)$

This section looks at one example to analyse the structure of the $p(a, z, t)$. Let

$$\begin{aligned}&[c_1, c_2, c_{0b}, c_{0w}, c_{ab}, c_{aw}, c_{tb}, c_{tw}]\\&= [0.83,\ 1,\ 2,\ 2.30,\ -30.19,\ 2.68,\ -22.52,\ 2]\,.\end{aligned}$$

We then plot

$$\begin{aligned}p(a, w, t) &= (2.30 + 2.68a + 2t)\, e^{-0.83t-a},\\ p(a, b, t) &= (2 - 30.19a - 22.52t)\, e^{-0.83t-a}.\end{aligned}$$

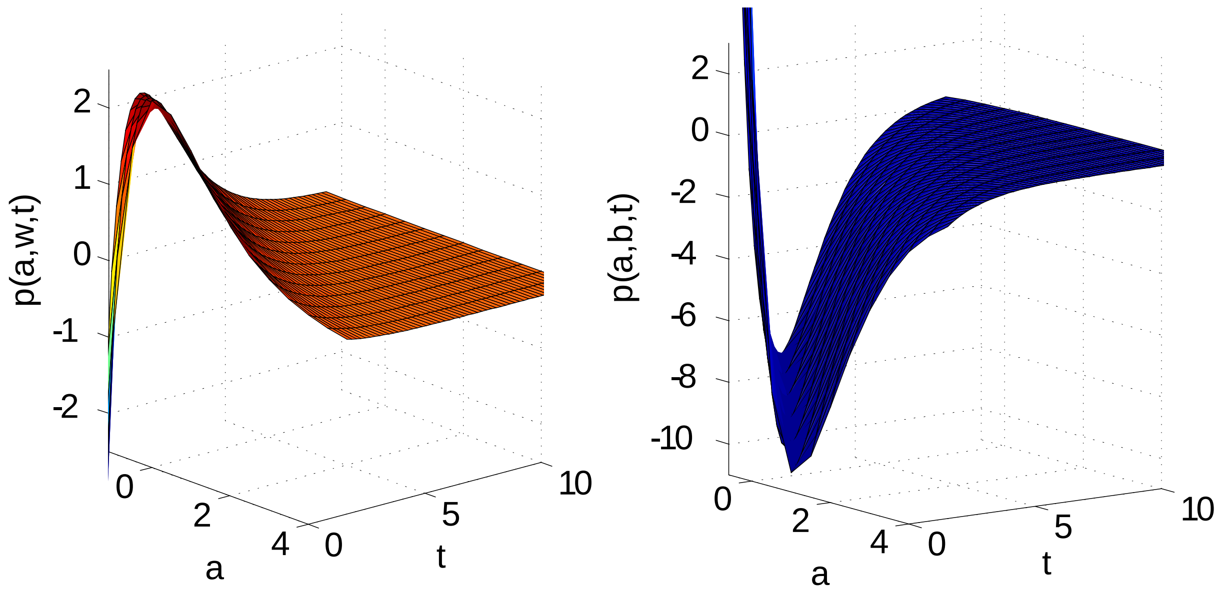

Figure 4 *$p(a, w, t)$ and $p(a, b, t)$ for $-1.5 \leq a \leq 5$ and $0 \leq t \leq 10$.*

To get some impression, we solve the system for $-1.5 \leq a \leq 5$ and $0 \leq t \leq 10$. We observe that the peak moves to the left as time goes by. Taking a bigger interval for a, this movement would of course become even more visible.

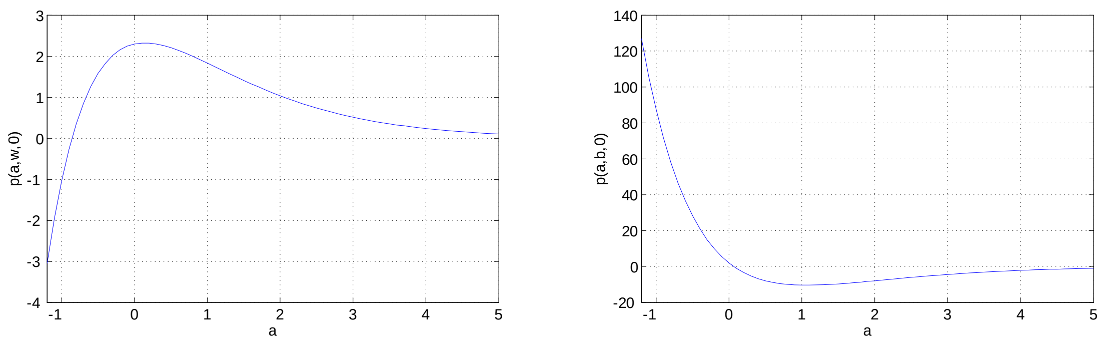

Figure 5 *Initial conditions $p(a, w, 0)$ and $p(a, b, 0)$ for $-1.5 \leq a \leq 5$.*

As any solution of a PDE is a function of the intitial condition, finding a closed-form solution implies in our case an explicit functional form for the initial condition (i.e. for $t = 0$). These initial conditions are shown in fig. 5.

4 Conclusion

We showed that for a special two-dimensional linear system of PDEs,

$$\frac{\partial}{\partial t}p(a,w,t) - m_w \frac{\partial}{\partial a}p(a,w,t) + sp(a,w,t) - \mu p(a,b,t) = 0,$$
$$\frac{\partial}{\partial t}p(a,b,t) - m_b \frac{\partial}{\partial a}p(a,b,t) + \mu p(a,b,t) - sp(a,w,t) = 0,$$

a solution of the form

$$p(a,z,t) = (c_{0z} + c_{az}a + c_{tz}t)\, e^{-c_1 t - c_2 a}, \qquad z \in \{w,b\}$$

exist, if the parameters fulfil certain conditions. In total we get five conditions leaving us a high degree of freedom by choosing three parameters.

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A Appendix - Proof of the main result

This section proves proposition 1. Computing the partials derivatives of

$$p(a,z,t) = (c_{0z} + c_{az}a + c_{tz}t)\, e^{-c_1 t - c_2 a},\ z \in \{w, b\}$$

and substituting those derivatives back into (1) yields

$$\begin{aligned}
0 &= c_{tw} - c_{aw}m_w + c_{0w}(-c_1 + c_2 m_w + s) - c_{0b}\mu \\
&\quad + (c_{tw}(-c_1 + c_2 m_w + s) - c_{tb}\mu)t + a(c_{aw}(-c_1 + c_2 m_w + s) - c_{ab}\mu), \qquad (9) \\
0 &= c_{tb} - c_{ab}m_b + c_{0b}(-c_1 + c_2 m_b + \mu) - c_{0w}s \\
&\quad + (c_{tb}(-c_1 + c_2 m_b + \mu) - c_{tw}s)t + a(c_{ab}(-c_1 + c_2 m_b + \mu) - c_{aw}s), \qquad (10)
\end{aligned}$$

where we already used the fact that $e^{-c_1 t - c_2 a} > 0$. If we can ensure that those two equations hold, $p(a,z,t)$ are solutions.

Due to the fact that (9) and (10) have to hold for all $t \in [0, \infty)$ and $a \in \mathbb{R}$, we need that

$$\begin{aligned}
c_{tw} - c_{aw}m_w + c_{0w}C_w &= c_{0b}\mu, && (11a) \\
c_{tw}C_w &= c_{tb}\mu, && (11b) \\
c_{aw}C_w &= c_{ab}\mu, && (11c) \\
c_{tb} - c_{ab}m_b + c_{0b}C_b &= c_{0w}s, && (11d) \\
c_{tb}C_b &= c_{tw}s, && (11e) \\
c_{ab}C_b &= c_{aw}s, && (11f)
\end{aligned}$$

where we simplified the expressions using $C_w = -c_1 + c_2 m_w + s$ and $C_b = -c_1 + c_2 m_b + \mu$.

Substitution of the second and third equation into the last two equations yields

$$\begin{aligned}
c_{tw} - c_{aw}m_w + c_{0w}C_w &= c_{0b}\mu, \\
\frac{c_{tw}}{c_{tb}} &= \frac{\mu}{C_w}, \\
\frac{c_{aw}}{c_{ab}} &= \frac{\mu}{C_w}, \\
c_{tb} - c_{ab}m_b + c_{0b}C_b &= c_{0w}s, \\
\frac{C_b}{s} &= \frac{\mu}{C_w}, \\
\frac{C_b}{s} &= \frac{\mu}{C_w}.
\end{aligned}$$

and hence we can eliminate one equation. The remaining system is

$$\begin{aligned}
c_{tw} - c_{aw}m_w + c_{0w}C_w &= c_{0b}\mu, \\
\frac{c_{tw}}{c_{tb}} &= \frac{\mu}{C_w}, \\
\frac{c_{aw}}{c_{ab}} &= \frac{\mu}{C_w}, \\
c_{tb} - c_{ab}m_b + c_{0b}C_b &= c_{0w}s, \\
\frac{C_b}{s} &= \frac{\mu}{C_w}. \qquad (12)
\end{aligned}$$

Therefore we have five equations for eight unknowns, yielding an underdetermined system.

Assume c_2 is free, i.e. let $c_2 \in \mathbb{R}\backslash\{0\}$. Now c_1 can be chosen such that the last equation, i.e.

$$\frac{-c_1 + c_2 m_b + \mu}{s} = \frac{\mu}{-c_1 + c_2 m_w + s} \tag{13}$$
$$\iff c_1^2 - c_1\left(\mu + s + c_2(m_b + m_w)\right) + c_2(c_2 m_b m_w + \mu m_w + s m_b) = 0$$

holds. Hence c_1 can be determined by solving a quadratic equation and we could end up with up to two solutions for c_1.Nevertheless we assume, that we can find at least one value for c_1 (maybe we have to change the chosen value of c_2) and therefore we have fixed c_1, c_2 and of course C_b, C_w.

The remaining four equations

$$c_{tw} - c_{aw} m_w + c_{0w} C_w = c_{0b}\mu, \tag{14a}$$
$$\frac{c_{tw}}{c_{tb}} = \frac{\mu}{C_w}, \tag{14b}$$
$$\frac{c_{aw}}{c_{ab}} = \frac{\mu}{C_w}, \tag{14c}$$
$$c_{tb} - c_{ab} m_b + c_{0b} C_b = c_{0w} s, \tag{14d}$$

can be used to determine four out of the remaining six parameters, $c_{0w}, c_{aw}, c_{tw}, c_{0b}, c_{ab}$ and c_{tb}. To start with, we look at the first and last equation. This system can be written in matrix notation as

$$\begin{pmatrix} c_{tw} - c_{aw} m_w \\ c_{tb} - c_{ab} m_b \end{pmatrix} = \begin{pmatrix} \mu & -C_w \\ -C_b & s \end{pmatrix} \begin{pmatrix} c_{0b} \\ c_{0w} \end{pmatrix} \equiv M \begin{pmatrix} c_{0b} \\ c_{0w} \end{pmatrix}.$$

Looking at $\det(M)$, we see that $\det(M) = \mu s - C_w C_b = 0$ by (12). Hence, we have either none or infinite solutions. Looking back at

$$c_{tw} - c_{aw} m_w = -c_{0w} C_w + c_{0b}\mu, \tag{15}$$
$$c_{tb} - c_{ab} m_b = c_{0w} s - c_{0b} C_b, \tag{16}$$

we can write those two equations as

$$c_{0w} = -\frac{c_{tw} - c_{aw} m_w}{C_w} + \frac{\mu}{C_w} c_{0b}, \tag{17}$$
$$c_{0w} = \frac{c_{tb} - c_{ab} m_b}{s} + \frac{C_b}{s} c_{0b}. \tag{18}$$

Finding a solution of a system of linear equations is equivalent with finding an intersection of those two lines. By comparing those two lines with each other, we see, that they have to be parallel. as $\frac{\mu}{C_w} = \frac{C_b}{s}$, according to (12). We therefore either have no solution or an infinite amount of solutions for (15) and (16). The second case - the one we need for our proof to work - arises for

$$-\frac{c_{tw} - c_{aw} m_w}{C_w} = \frac{c_{tb} - c_{ab} m_b}{s}. \tag{19}$$

We therefore need to add this condition to our list of the "remaining four equations" in (14).

Doing so, we can remove one of the two redundant equations (given that we impose them to be identical in (19)) and obtain

$$\frac{c_{tw}}{c_{tb}} = \frac{\mu}{C_w},$$
$$\frac{c_{aw}}{c_{ab}} = \frac{\mu}{C_w},$$
$$c_{0w} = \frac{c_{tb} - c_{ab} m_b}{s} + \frac{C_b}{s} c_{0b},$$
$$-\frac{c_{tw} - c_{aw} m_w}{C_w} = \frac{c_{tb} - c_{ab} m_b}{s}.$$

We start by solving the first two equations for c_{ab} and c_{tb} respectively and insert those expression into the third equation. This yields

$$c_{tw}\frac{C_w}{\mu} = c_{tb}, \tag{21}$$

$$c_{aw}\frac{C_w}{\mu} = c_{ab}, \tag{22}$$

$$-sc_{tw} + sm_w c_{aw} = c_{tw}\frac{C_w^2}{\mu} - c_{aw}\frac{C_w^2 m_b}{\mu},$$

where the last equation is equivalent to

$$c_{aw}\left(sm_w + \frac{C_w^2 m_b}{\mu}\right) = c_{tw}\left(\frac{C_w^2}{\mu} + s\right)$$

$$\iff c_{aw} = c_{tw}\left(\frac{C_w^2 + s\mu}{\mu}\right)\left(\frac{sm_w\mu + C_w^2 m_b}{\mu}\right)^{-1}$$

$$\iff c_{aw} = c_{tw}\left(\frac{C_w^2 + s\mu}{sm_w\mu + C_w^2 m_b}\right). \tag{23}$$

Let $c_{tw} \in \mathbb{R}\backslash\{0\}$ be fixed but arbitrary, then (23) determines c_{aw}. Also c_{ab} and c_{tb} can now be computed using (21) and (22).

At this stage we are left with only one equation,

$$c_{0w} = -\frac{c_{tw} - c_{aw}m_w}{C_w} + \frac{\mu}{C_w}c_{0b}, \tag{24}$$

to determine c_{0w} and c_{0b}. Without loss of generality, we assume c_{0b} to be the third and last free parameter and with (24) we have the condition on the missing parameter c_{0w}.Therefore the five conditions of theorem 1 are (12), (21), (22), (23) and (24).

q.e.d.