Journal of Economic Theory **87**, 194–217 (1999)
Article ID jeth.1999.2529, available online at http://www.idealibrary.com on IDEAL®

Optimal Saving under Poisson Uncertainty*

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Received November 5, 1997; revised February 15, 1999

This paper studies optimal saving and investment behaviour of a household that can either invest in a riskless or a risky saving technology when risk results from a Poisson process. The focus is on behaviour of households in a general equilibrium setup. Poisson processes are introduced since they allow us to understand and model endogenous cycles jointly with long-run growth. It turns out that a very simple, intuitive and tractable characterization of equilibrium is possible. *Journal of Economic Literature* Classification Numbers: C61, O33, O41. © 1999 Academic Press

1. INTRODUCTION

Everyday life is characterized by uncertainties of many kinds. Investing in stocks yields uncertain returns, doing research does not always lead to predictable results and looking for a particular good, a particular person or a better job usually takes an unknown length of time. Examples of the type of uncertainty the present paper is concerned with are research projects or search processes in general. When searching for, e.g., a job, the outcome of this search per day or week is to either find one or not. When undertaking R&D for a new technology where it is unclear how long it will take to develop this technology, each particular day of this research venture is characterized by discovery of the new technology or not. Hence, in contrast to, e.g., investment into stocks where returns can take (almost) any value, uncertainty in search processes is resolved in one of two ways: searching is either successful or not.

* This paper was written while working at the University of Dortmund, Germany. I thank Peter Bank, Roland Demmel, Heinz Holländer, Christian Kleiber, Bernhard Nietert, Sven Rady, and YEDL (The Young Economists' Discussion List (www.uni-dortmund.de/YEDL)) for discussions and comments and Chuck Humphreys for support. I am indebted to an associate editor for very helpful and constructive comments. Views expressed in this paper do not necessarily represent the views of the World Bank, its Executive Directors, or the countries they represent. Financial support by the Graduiertenkolleg "Allokationstheorie, Wirtschaftspolitik und kollektive Entscheidungen" of the Deutsche Forschungsgemeinschaft is gratefully acknowledged.

0022-0531/99 $30.00

Uncertainties of the latter type are often studied in economics and they are frequently modelled as Poisson processes. Matching models of the Diamond–Mortenson–Pissarides type are one example, investment into research in the new growth theory literature based on Aghion and Howitt [1] or the Grossman and Helpman [7] type quality ladder models is another. Questions that arise in this context concern the optimal consumption pattern, the optimal amount of resources used for searching or the optimal amount of investment into research.

Despite the popularity of Poisson-based models, only Merton [11] has studied optimal saving decisions of a risk-averse household when the household's budget constraint contains some Poisson distributed stochastic variables. In his partial equilibrium perspective, he has focused on closed form solutions (cf. also Sethi and Taksar [14]) as he assumed that some variables that are not central to his question are constant. Other authors have generally circumvented a detailed study of the optimization problem of households by assuming either risk-neutrality of households or complete diversification of risk on the aggregate level.

The present paper fills this gap and studies optimal consumption and investment decisions of risk-averse households with a general equilibrium setup in mind. The assumption of constant time paths for some variables is then no longer appropriate and it becomes advisable to rather work with a Keynes–Ramsey rule. It turns out that, under standard assumptions concerning the economic environment of the household, optimal household behaviour can be described in an alternative way that is more tractable for general equilibrium analysis: either the Keynes–Ramsey rule will take its deterministic form or a "consumption jump condition" holds that prescribes the jump of consumption whenever the Poisson process has a positive increment.

This paper is part of the economics literature on optimal control of continuous-time stochastic processes. Apart from Merton's paper mentioned above, this literature has almost exclusively focused on diffusion processes. Examples are Eaton [3], Turnovsky [16] and Obstfeld [12]. An overview of the literature and techniques is provided by Dixit and Pindyck [2] and by Turnovsky [17, part IV].

The motivation for this paper stems from the idea that including jump processes, of which the Poisson process is the most prominent example, in general equilibrium models allows us to provide a mechanism that leads to endogenous business or Kondratieff cycles in a very natural way. The usefulness of this approach has been shown in a couple of recent papers. Wälde [18] studies optimal business cycles in a Poisson–Ramsey centrally planned growing economy with homogenous capital. Wälde [19] analyses the endogeneity of cycles in an imperfect competition framework where firms choose whether to invest in refinement of a given technology or in technological breakthroughs. Finally, Wälde and Walz [21] study the question to what extent business cycles are too long. Is the length of time between two peaks of a cycle too long or too short?

The consumer maximization problem in these papers is similar. The present paper therefore presents a prototypical consumer maximization problem which can directly be applied for modelling utility maximization in decentralized economies. The derivation of optimality conditions in the present paper is more detailed than in the just-mentioned studies. This paper also contains a complete appendix which is not included in the other papers.

The setup of the paper is as follows. The next section starts by presenting saving technologies. This is the only information about the economic environment needed to solve the household's maximization problem. We do not spell out a complete general equilibrium model as this is beyond the scope of the present paper. Section 2 also presents household preferences and derives the reduced form of the household optimization problem and the corresponding Bellman equation. Section 3 derives and interprets first order conditions that determine the optimal consumption level and the optimal allocation of savings to different investment forms. These first order conditions are then used to derive the Keynes–Ramsey rule. Section 4 puts the household in a general equilibrium context. Two common assumptions about the structure of the economy make the model more tractable. A theorem will be proven that describes optimal investment behaviour in general equilibrium. A consumption jump condition is derived that determines the jump in consumption when an increment in the Poisson process takes place. Section 5 presents informally general equilibrium models that use the framework presented here and thereby highlights the economic significance of the results. The last section concludes. The Appendix contains several derivations, including the derivation of the Keynes–Ramsey rule and various proofs.

2. THE MODEL

2.1. *Saving Technologies*

Imagine an economy with two saving technologies. Think, as an example, of firms that can increase output by either buying more machines of the same type or by developing a more productive machine. Or think of firms that can imitate existing products or develop a new product. It can be easily imagined that these activities differ in their degree of riskiness. Buying more machines of the same type, i.e., accumulating capital, is a fairly riskless activity compared to developing new production units where basically new technologies have to be found. Similarly, imitating existing products is less risky than developing a new product. This difference in riskiness between different saving forms has already been stressed by Shell [15] though no explicit analysis of optimal behaviour in such a risky environment has been undertaken.

Let us here make the extreme assumption that one saving technology is riskless while the other saving technology is risky. Using for the sake of concreteness capital accumulation as an example, employing investment goods K and other resources R for the riskless activity implies that the stock of capital K increases by a certain amount dK per unit of time dt,

$$dK = A^{\gamma} F(K, R)\, dt, \tag{1}$$

where A is constant productivity parameter, A^{γ} is total factor productivity and $F(\cdot)$ is a "standard" production function.

By contrast, spending I on resources for risky R & D leads to a development of a new production unit and thereby to capital accumulation only with a certain probability. Once this prototype unit is developed, however, capital in (1) can be produced with a higher total factor productivity and subsequent capital accumulation is more efficient. This type of uncertainty is conveniently modelled by a Poisson process and R & D can be formalized as

$$d\gamma = dq, \tag{2}$$

where dq is the increment of the Poisson process q. With this formulation, the process q counts the increases in the technology level. Hence, whenever research is successful, q increases by one ($dq = 1$) and total factor productivity of investment goods in (1) increases multiplicatively by A[1] as γ rises by one. In addition, the economy's aggregate capital stock rises by $\tilde{K}$ which represents the prototype unit that comes out of the R & D process. Both saving forms therefore eventually increase the stock of capital but only the risky one increases total factor productivity. The state of the economy at each point in time is consequently determined by a capital stock K and a level of technology γ.

The probability per unit of time dt that research is indeed successful, i.e., that $dq = 1$, is given by $\lambda\, dt$, where λ is the arrival rate of the Poisson process. Let this arrival rate depend on the amount of resources I spent on R & D,

$$\lambda = \lambda(I),$$

[1] It is clear from this formulation that one could equivalently express total factor productivity by A^q without having to use the variable γ.

with $\lambda'(I) > 0$. The more that is spent per unit of time on research, the higher the probability that a new technology is actually found. For more details and a more formal introduction to Poisson processes, cf., e.g., Ross [13] or Kingman [9].

In what follows, we will adopt a linear specification for $\lambda(I)$ as this is the specification that allows us to understand the research sector as a sector that works under perfect competition. Hence, the analogy to standard constant returns to scale we adopt here for the arrival rate is

$$\lambda = bIp_I^{-1}, \tag{3}$$

where b is a constant productivity parameter and p_I is the price of a unit of resources used for R & D.

Apart from the development of a better production unit and the associated productivity increase, we allow for a further effect of successful R & D on the value of the economy's capital stock. This is motivated by Schumpetarian creative destruction mechanisms, as modelled, e.g., by Aghion and Howitt [1] and many others, which affect the economy through a price and a quantity channel: A new technology changes the value v of a unit of capital either because capital becomes more productive or because the market shares of firms change as a competitor has a better technology. A new technology also changes the physical capital stock in the economy as a certain share $0 \leqslant s_k \leqslant 1$ of production units might become obsolete when firms are driven out of the market. Clearly, the precise mechanisms depend on general equilibrium effects that are not modelled here. Summing up the effects on the economy's aggregate capital stock, its change after successful R & D is given by

$$\Delta K = \tilde{K} - s_k K. \tag{4}$$

Due to this stochastic depreciation, holding capital is not riskless, despite the deterministic accumulation technology (1).

2.2. *Consumers*

Consider a household that owns assets whose value a is given by $a = kv$, the product of the capital stock k owned by the household and the value v of one unit of capital. Current household income consists of interest payments on those assets and wage income, $ra + w$, and current household expenditure consists of R & D expenditure i and consumption expenditure e. The difference between current income and total expenditure equals current savings that in- or decrease assets held by the household.

Asset holdings of a household also change when R & D is successful. In this case, the household receives a certain share of the total value ϖ of the new production unit $\tilde{K}$ in (4) that resulted from the successful R & D project. While ϖ is exogenous to the household, this share is assumed to be determined by a simple sharing rule: Each household obtains the share that is given by its share in total expenditure I for this project, $\varpi i/I$. On the other hand, it loses a certain share $s \leqslant 1$ of its assets due to the price channel that affects v and the quantity channel that renders the share s_k of physical capital (4) obsolete. Both channels are summarized in the parameter s that is exogenous to the household.

Note that the share s can be zero or even negative. The latter would mean that the creative effect of creative destruction is larger than the destructive effect and, overall, the value of a household's capital stock $\tilde{k}$ after successful R & D is larger than the value of the household's capital stock k before successful R & D.

A household in such an environment faces a decision problem which consists of two choices: How much do they want to save, respectively consume, and how much of their savings do they want to spend on R & D? Modelling individual investment i in R & D as current expenditure, the budget constraint reads

$$da = (ra + w - i - e)\, dt + \left(\varpi \frac{i}{I} - sa \right) dq. \tag{5}$$

While a formal derivation, starting from properties of technologies (1) and (2) above, is in Appendix 1, an intuitive understanding is as follows: The budget constraint is given by a stochastic differential equation that consists of a deterministic and a stochastic part. The first part on the right-hand side (RHS) is the deterministic part which says that household wealth a increases per unit of time dt by an amount that is given by current savings, $ra + w - i - e$. The interest rate r captures all instantaneous changes in the value v of capital and possibly dividend payments.

The second, stochastic, part is governed by the increment dq of the above Poisson process. It reflects changes in household wealth as a function of the outcome of the R & D project that affect both changes in the value v and changes in quantities k. When no innovation takes place, i.e., when $dq = 0$, the budget constraint has, after dividing by dt, a well-known form. When the research project is successful, $dq = 1$, assets of the household are reduced by sa, but, at the same time, increase by payments from cofinancing successful R & D. Hence, the stock of assets after an innovation is given by[2]

$$\tilde{a} = \varpi \frac{i}{I} + (1 - s)a.$$

[2] Note the similarity to the change in the aggregate physical capital stock (4). Cf. also Appendix 1.

Let the household's planning horizon be infinite. Expected utility of a household at time t is given by

$$U(t) = \mathbf{\varepsilon} \int_{t}^{\infty} e^{-\rho[\tau - t]} u(c(\tau))\, d\tau, \tag{6}$$

i.e., the expected sum of discounted future instantaneous utility flows $u(c(\tau))$, where ρ is the time preference rate, $c(\tau)$ are consumption flows and $\mathbf{\varepsilon}$ denotes the expectations operator.

2.3. *The Optimization Problem*

We now consider a household that maximizes (6) subject to the budget constraint (5) using as control variables instantaneous consumption expenditure e and research expenditure i and having as state space (a, γ). Aggregate variables as the consumption good price p, the interest rate r, the wage rate w and the arrival rate λ are considered to be exogenous to the household. Consumption expenditure is linked to consumption c by $e = pc$.

This is a standard consumption and investment choice problem; the novel aspect is the type of uncertainty, i.e., Poisson uncertainty. It is solved by dynamic programming methods which imply the Bellman equation (cf. Appendix 2)

$$\rho V(a, \gamma) = \max_{e, i} \{u(c) + V_a(a, \gamma)[ra + w - i - e] + \lambda(I)[V(\tilde{a}, \gamma + 1) - V(a, \gamma)]\}. \tag{7}$$

It says that the return per unit of time of holding assets a and given the technology level γ is given by the sum of instantaneous returns from holding these assets, i.e., instantaneous utility flow $u(c)$, and the change in the value of this asset. This change is twofold: First, the value changes because the stock of assets changes as a result of accumulation of assets. This is the second term on the RHS, $V_a(a, \gamma)[ra + w - i - e]$, which is the marginal value of an additional unit of assets, the partial derivative $V_a(a, \gamma)$, times total savings allocated to asset accumulation. Second, it changes in probability by the product of the arrival rate λ and the change in the value of holding assets a if R & D is successful. This change equals the difference between the value of holding a stock of assets $\tilde{a}$ in an economy with a technology of vintage $\gamma + 1$ (i.e., after an R & D success) and holding the current stock a under vintage γ.

The difference $V(\tilde{a}, \gamma + 1) - V(a, \gamma)$ shows that incentives to invest in R & D come from two sources: First, the change in the level of assets from a to $\tilde{a}$ (which might actually be a disincentive if $\tilde{a} < a$) and second, the change in the value $V(\cdot)$ of holding assets due to the change in total factor productivity. Clearly, the latter can be expected to be much weaker than the former.

3. FIRST ORDER CONDITIONS AND KEYNES–RAMSEY RULE

The first order condition for expenditure e reads[3]

$$u'(c)\, p^{-1} = V_a(a). \tag{8}$$

Losses in instantaneous utility due to less expenditure have to be matched by gains in future utility captured by the marginal value of an additional unit of assets. The left-hand side (LHS) is the derivative of instantaneous utility from consumption $u(c)$ with respect to consumption expenditure e. Equation (8) is familiar from first order conditions under certainty.

The first order condition for investment expenditure i reads

$$V_a(a) = \lambda V_{\tilde{a}}(\tilde{a}) \frac{d\tilde{a}}{di}. \tag{9}$$

When deciding where to allocate one's savings, a household trades off gains from an additional invested unit in certain asset accumulation and expected gains from spending an additional unit for uncertain R & D projects. The LHS is the gain from investing an additional unit in accumulation of certain capital, given by the marginal value of a unit of assets. This must equal the gains from an additional unit of investment in R & D projects on the RHS. This is given by the arrival rate times the marginal value of assets $\tilde{a}$ when technology is of vintage $\gamma + 1$ times the change in assets as i changes.

Note that in deriving this condition, we assume, as stressed before, that individuals neglect the effect their R & D finance decision has on the arrival rate λ of an innovation. Practically speaking, when supplying venture capital, individuals take the riskiness of the R & D process as given and neglect their individual impact on it. This is in analogy to pricetakership in goods or labour markets[4].

[3] In what follows, we simplify notation by writing $V(a)$ instead of $V(a, \gamma)$ and $V(\tilde{a})$ instead of $V(\tilde{a}, \gamma+1)$. As $\tilde{a}$ determines the wealth stock once the new technology is available, no ambiguity arises. Further, λ replaces $\lambda(I)$.

[4] None of the results in this paper depend on this assumption. As was shown in an earlier Discussion Paper version, which is available at http://www.uni-dortmund.de/YEDL/pblicatn.html, the same results hold. The present approach is realistic and considerably simplifies the exposition.

The Keynes–Ramsey rule is obtained by starting from the maximized Bellman equation which has the same form as (7) only that control variables are now functions of the stock of assets a. Computing the differential with respect to a and inserting first order conditions gives (cf. Appendix 3)

$$[\rho - r + \lambda]\, V_a(a) - [1-s]\, \lambda V_{\tilde{a}}(\tilde{a}) = V_{aa}(a)[ra + w - i - e], \qquad (10)$$

an equation with close similarities to deterministic models. Since the shadow price of holding an asset is given by $V_a(a)$, the first term on the LHS is identical (neglecting λ) to standard conditions in deterministic problems. The RHS is the equivalent of the time derivative of the shadow price of the state variable in deterministic problems. In fact, with $i = 0$, i.e., when all savings are allocated to asset accumulation, $\lambda = 0$ as well and (10) just equals the condition for the evolution of the costate variable in a standard Ramsey growth model. The second term on the LHS is novel and becomes relevant when not all old assets (in which case $s = 1$) are devaluated to zero after successful R & D.

This equation can be reformulated by using the budget constraint (5) and the first order conditions (8) and (9). This gives the Keynes–Ramsey rule for optimal allocation of consumption over time (cf. Appendix 4),

$$-\frac{u''(c)}{u'(c)}\, dc = \left[r - \frac{dp/dt}{p} - \lambda - \rho \right] dt + [1-s]\, \lambda \Omega\, dt - \frac{u''(c)}{u'(c)}\, c_a \left[\varpi \frac{i}{I} - sa \right] dq, \qquad (11)$$

where

$$\Omega = \frac{u'(c(\tilde{a}))\, p}{u'(c(a))\, \tilde{p}}.$$

While p denotes as before the current price of the consumption good, $\tilde{p}$ stands for the price of the consumption good after the development of a new technology. The deterministic part of this equation has similarities to "standard" Keynes–Ramsey rules. One finds, for example, the difference between the inflation-adjusted interest rate and the time preference rate. A first difference to standard rules is that the real interest rate is reduced by the arrival rate λ. This reminds us of the well-known property of Poisson uncertainty which increases the discount factor in models of imitation (cf., e.g., Grossman and Helpman [6]): When assets can be devaluated, a household is willing to save (which leads to rising consumption levels) only if the interest rate lies above the time preference rate to such an extent that the probability of becoming devaluated is accounted for.

A second difference to standard rules is the partial correction of the increase in impatience by the last term $[1-s]\, \lambda \Omega$ which depends on the share s of assets which lose their value, the arrival rate λ and Ω. The variable Ω stands for the ratio of marginal utility from an additional unit of expenditure under vintage $\gamma + 1$ and with asset holdings $\tilde{a}$ and marginal utility under the current technology γ and asset holdings a. The expected value is given by the product of the arrival rate λ and Ω. The extent to which the prospect of a new technology increases consumption growth depends on the share s of old assets that are not devaluated under the new technology. If all assets are devaluated, $s = 1$, the entire effect vanishes. The fewer assets lose their value, i.e., the lower s, the higher this correction.

The stochastic part gives the jump in consumption as a result of a success in the R & D project, i.e., when $dq = 1$. For the instant the jump takes place, the above rule reads (set $dt = 0$ and $dq = 1$) $dc = c_a[\varpi i I^{-1} - sa]$: the discrete change in consumption equals the partial derivative of consumption with respect to asset holdings times the change in asset holdings.

4. CORNER SOLUTIONS IN GENERAL EQUILIBRIUM

The last section has derived the Keynes–Ramsey rule for households facing Poisson uncertainty. It has been assumed implicitly that interior solutions for both first order conditions (8) and (9) exist. We can obtain further insight into the behaviour of households in a general equilibrium setup by assuming a representative household. It will then become clear why corner solutions (all savings will be allocated either to R & D investment or to asset accumulation) are quite common and interior solutions for (9) do generally not exist. It will further be shown that resulting optimality conditions are more easy to use in general equilibrium models than the above Keynes–Ramsey rule.

4.1. *Corner Solutions*

Assuming a representative household and a linear arrival rate (3) in general equilibrium leads to corner solutions. Households either spend all of their savings on R & D or in accumulating capital. The first assumption is standard in representative agent models where the focus is on aggregate aspects only. If households differed in their initial asset endowment, they would spend different amounts i on R & D. This would lead to heterogeneous accumulation paths for assets which would be very interesting to study but which are not the scope of the present paper. The second assumption of constant returns to scale is natural as long as one does not want to focus on the behaviour of imperfectly competitive firms in the R&D sector.

This bang-bang behaviour might appear surprising since one is used to at least some diversification of agents. An agent that can choose to invest savings either in a riskless or a risky asset will generally allocate some share of his investment into the risky investment form. The same behaviour can be observed here at the household level when looking at the first order condition for shares in (9). Given the assumption of identical agents, however, relative investment of a household in R & D projects is constant, $i/I = m^{-1}$, where m is the number of decision units in this economy. This implies that—de facto—an individual no longer has a continuous choice of how much of her savings to invest into the risky asset but it becomes a discrete choice. An individual either obtains the value associated with a share m^{-1} of dividend payments or nothing. Therefore, the individually continuous choice is a discrete choice in general equilibrium which implies corner solutions.

Formally, computing the derivative of the Bellman equation (7) with respect to i and inserting into this derivative the representative agent implication $i/I = m^{-1}$ and the linear arrival rate (3) gives

$$\frac{d}{di}\{u(c) + V_a(a)[ra + w - i - e] + \lambda[V(\tilde{a}) - V(a)]\} = -V_a(a) + b\frac{\varpi}{p_I}V_{\tilde{a}}(\tilde{a}), \tag{12}$$

where

$$\tilde{a} = \varpi\frac{1}{m} + (1-s)a.$$

This shows that the decision to invest in R & D or not is indeed a zero-one decision. Given a stock of assets a before and a stock of assets $\tilde{a}$ after innovation and a constant number of decision units m, this derivative equals zero only for one specific a^*. In short, the Bellman equation (9) is de facto linear[5] in an individual's investment expenditure i and displays familiar bang-bang behaviour.

We will now derive an investment rule for households that says under what conditions to use savings for capital accumulation and under what conditions to use savings for R & D. Before providing a theorem, we consider the following

LEMMA. *If initial conditions are such that $i = 0$, i.e., if the aggregate capital stock K is sufficiently low relative to the technology level γ,*

$$i = \left\{\begin{matrix} 0 \\ ra + w - e \end{matrix}\right\} \Leftrightarrow b\frac{\varpi}{p_I}V_{\tilde{a}}(\varpi m^{-1} + (1-s)a)\left\{\begin{matrix} < \\ = \end{matrix}\right\} V_a(a) \tag{13}$$

[5] The Bellman equation is non-linear in i. Taking the representative consumer aspect into consideration makes the derivative of the Bellman equation (with respect to i) independent of i, *as if* the Bellman equation was linear in i. Hence, the term de facto linear.

and V_a can never be strictly smaller than $b\varpi p_I^{-1} V_{\tilde{a}}(\tilde{a})$. In words, all savings are spent on capital accumulation, $i=0$, when the marginal value of an additional unit of capital obtained through capital accumulation, $V_a(a)$, is higher than the expected gain from several additional units, $b(\varpi/p_I)\, V_{\tilde{a}}(\tilde{a})$. All savings are used for R & D, $i = ra + w - e$, when certain returns equal expected returns.

Proof (cf. Appendix 5). This lemma is now used to prove the following.

THEOREM. *Under the same assumptions as in the above lemma,*

$$i = \begin{Bmatrix} 0 \\ ra + w - e \end{Bmatrix} \Leftrightarrow r - \rho - \lambda[1 - [1 - s]\Omega] \begin{Bmatrix} > \\ = \end{Bmatrix} 0. \tag{14}$$

Proof (cf. Appendix 6).

This theorem also gives a rule about where to allocate one's resources, but, in contrast to the above lemma where value functions are part of the condition, the condition where to invest is expressed in terms of known functions: When the difference between the interest rate, the time preference rate and the adjusted arrival rate is positive, all savings are used for capital accumulation. When (14) holds as an equality, all savings are channelled into R & D.

4.2. *The Consumption Jump Condition*

Given this investment rule (14), the behaviour of households is described in either of two ways: When no R & D is financed, i.e., when $i = 0$, consumption growth is given by a familiar version of the Keynes–Ramsey rule (11) which reads

$$-\frac{u''(c)}{u'(c)}\,\dot{c} = r - \frac{\dot{p}}{p} - \rho. \tag{15a}$$

This rule holds simply because with zero investment in R & D, $I = 0$, the arrival rate is zero, $\lambda = 0$, and, consequently, the increment of the Poisson process as well, $dq = 0$. As $\lambda = 0$, this could be called the deterministic regime. The budget constraint (5) is given by

$$\dot{a} = ra + w - e. \tag{15b}$$

When all savings are allocated to investment in R & D, i.e., $i = ra + w - e$, each household's stock of asset is constant and so is his consumption level, given the consumption first order condition (8). This will be termed the stochastic regime[6]. The question now arises how consumption and asset stocks change between these two regimes.

At the moment when the economy moves from the deterministic to the stochastic regime, neither asset stocks nor the technology level γ change; therefore, by (8), consumption does not change either. When a new technology is found, i.e., when the economy moves from the stochastic regime to the deterministic regime, the stock of assets changes according to

$$da = \left[\varpi \frac{i}{I} - sa \right] dq, \tag{16}$$

which directly follows from the budget constraint; assets jump in the case of a successful research project. What is now needed is a rule that describes the behaviour of consumption between the stochastic and the deterministic regime. As consumption can be expected to jump, we will call this rule consumption jump condition.

This condition is derived by considering the derivative of the Bellman equation with respect to R & D expenditure i in (12). As the lemma has shown, households switch from capital accumulation to financing R & D when the marginal value of an additional unit of capital obtained through capital accumulation equals the expected gain from increasing the asset stock through R & D. At the moment of the switch (and thereafter as the economy enters a stationary state), the derivative in (12) equals zero. Using the one-to-one mapping from asset stocks (and technology levels γ) to consumption levels as given by the first order condition for consumption (8), inserting (8) into (9) (where (9) is identical to (12) when the latter equals zero) together with the linear arrival rate (3) gives the consumption jump condition

$$u'(c)\frac{1}{p} = b\frac{\varpi}{p_I}u'(\tilde{c})\frac{1}{\tilde{p}}. \tag{17}$$

Consumption jumps such that the marginal utility from expenditure "today" (before innovation) on the LHS equals expected marginal utility "tomorrow" (after innovation) when total dividend payments are ϖ.

Consider now two examples for this condition. Assume that the utility function of each household is of the CES form

$$u(c) = c^{\sigma}. \tag{18}$$

[6] Since all households have a budget constraint of this type, the economy finds itself in a (temporary) stationary state since the stock of assets (and therefore, on the aggregate level, the stock of capital) is constant.

The consumption jump condition (17) then reads

$$\frac{c^{\sigma-1}}{p} = \frac{b}{p_I}\frac{\tilde{c}^{\sigma-1}}{\tilde{p}}\varpi \Leftrightarrow \tilde{c} = \left(\frac{b}{p_I}\frac{p}{\tilde{p}}\varpi\right)^{\varepsilon} c, \tag{19a}$$

where $\varepsilon \equiv 1/(1-\sigma)$. Consumption jumps upwards if the term in brackets exceeds unity. The jump is larger, the higher the intertemporal elasticity of substitution ε. The amplitude of the jump is determined by the marginal arrival rate b, dividend payments ϖ, the relative price of the consumption good before and after the innovation and by the price of the investment good. The higher dividend payments, the lower the relative price and the cheaper the investment good, the higher the jump in consumption.

With a logarithmic utility function, the consumption jump condition (17) becomes

$$\frac{1}{cp} = \frac{b}{p_I}\frac{1}{\tilde{c}\tilde{p}}\varpi \Leftrightarrow \tilde{c} = \frac{b}{p_I}\frac{p}{\tilde{p}}\varpi c. \tag{19b}$$

A final remark on consumption levels. No analysis of household behaviour is complete, unless a condition is stated which fixes the initial consumption level. A two dimensional differential equation system requires two boundary values to predict unique paths. One initial condition here as elsewhere is the initial asset stock, given by history. The other condition is "usually" a transversality condition which assures that the aggregate economy reaches a steady state or the balanced growth path. Here, the same is true. A transversality condition should be added as a further necessary and sufficient condition for optimality of the chosen consumption paths. The central difference from standard analysis is the fact that the necessary condition for the evolution of consumption over time has been extended by jump conditions. The path for consumption is therefore no longer continuous but includes jumps.

5. APPLICATIONS

This section briefly discusses possible applications of the present analysis[7]. Poisson processes are frequently used in economics to model R & D or matching processes. Think of the new growth theory or of Diamond–

[7] For detailed elaborations cf. Wälde [18, 19] or Wälde and Walz [21].

Mortenson–Pissarides models of unemployment[8]. No analysis had been undertaken, however, that combines risk-aversion and Poisson uncertainty in general equilibrium (cf., however, Wälde [20]). The approach here allows to combine these two aspects and promises new insights into the link between growth and cycles.

Jovanovic and Lach [8] and Greenwood, Hercowitz and Krusell [5] provide empirical support for the cycle generating property of the introduction of new products or technologies. Using different methods, they found that between 25% and one third of fluctuations in US GDP (in the period 1887 to 1972 and 1954 to 1990, respectively) can be explained by new technologies. Aggregate uncertainty and fluctuations due to innovations on a firm or industry level therefore appears to be a plausible approach.

Using results obtained in the present paper, their findings can be modelled as follows: The R & D technology in (2) and (3) represents an industry or economy wide R & D process whose aggregate implications can not be completely diversified. When an innovation takes place that increases total factor productivity A^{γ} in (1) by rising γ to $\gamma+1$, the marginal product of capital increases and the interest rate r goes up. An innovation also leads to a jump of assets a of households (either up or down, depending on the magnitude of s and ϖ in (5)) and, as a consequence of the change in assets and the change in the interest rate, of consumption (either up or down, as well, as explained after (19a)).

Households will profit from these higher returns and will shift their savings towards capital accumulation. The relevant laws of motion are (15a, b) and the economy follows a deterministic path. With decreasing returns to capital (i.e., if r falls as the aggregate capital stock rises), this growth process peters out until, at some point when (14) holds with equality, R & D will start again and the economy is in its stochastic regime. As long as R & D is not successful, both the consumption level and household assets are constant. Once this R & D process leads to a new innovation, capital accumulation starts again. Hence, each new innovation causes a new cycle.

This is only the basic mechanism that causes business cycles or longer cycles in these models. The main advantage of models of this type is the endogeneity of the point in time when R & D starts. This allows to study the effects of various policy measures on, e.g., the length of business cycles or their amplitude. Additional interesting features enter the analysis by further exploring imperfect competition and creative destruction. The first firm to discover a new technology is a monopolist while over time imitators

[8] It should be noted that uncertainty in R & D stems from uncertain returns to investment while uncertainty in matching models stems from uncertain labour income which implies some differences in the formal structure of the maximization problem.

reduce the market power of the innovator. Distortions are therefore higher at the beginning of a cycle than towards the end.

6. CONCLUSION

This paper has studied optimal saving and investment decisions of a household that can invest in a riskless and a risky asset. The distinguishing feature of this study is that risk associated with the risky asset arises from a Poisson process. In the most general case, consumption evolves over time according to a Keynes–Ramsey rule.

When two very common assumptions are made—perfect competition in the R & D sector and the concept of a representative consumer—the model (and potential applications) become much more tractable. Under these assumptions, savings are either allocated to the risky or to the riskless asset. Optimal consumption is then governed by *three* optimality conditions. It follows a standard Keynes–Ramsey rule familiar from deterministic models for periods where no investment into R & D takes place. Consumption results from a static allocation problem in periods where investment in R & D takes place but R & D is not successful. The third optimality condition is a consumption jump condition. It becomes relevant after a new technology has been developed and determines the size of the jump of consumption.

Applying these results in general equilibrium models with an explicit production side allows to study the implication of market structure or policy measures on the length and the amplitude of business cycles at the same time as on aggregate growth.

APPENDIX 1

The Budget Constraint (5)

In order to be able to derive the household's budget constraint, we need the following version of Ito's lemma. Let $\mathbf{z} \equiv (z_1, z_2)^T$ be a vector-valued Poisson process consisting of two independent Poisson processes z_1 and z_2. Let $f(\mathbf{x}) \equiv (f_1(\mathbf{x}), f_2(\mathbf{x}))^T$, $g(\mathbf{x})$, and $\sigma(\mathbf{x}) \equiv (\sigma_1(\mathbf{x}), \sigma_2(\mathbf{x}))^T$ be continuous functions having as argument the vector $\mathbf{x} \equiv (x_1, x_2)$. Functions f_i, g and s_i $(i=1, 2)$ are mappings from R^2 to R. Let $\mathbf{x}$ follow $d\mathbf{x} = f(\mathbf{x})\, dt + \sigma(\mathbf{x})\, d\mathbf{z}$. Following Gihman and Skorohod [4, ch. II.2.6], the differential $dg(\mathbf{x})$ is given by

$$dg(\mathbf{x}) = [g_{x_1}(\mathbf{x})\, f_1(\mathbf{x}) + g_{x_2}(\mathbf{x})\, f_2(\mathbf{x})]\, dt + [g(x_1 + \sigma_1(\mathbf{x}), x_2) - g(\mathbf{x})]\, dz_1 + [g(x_1, x_2 + \sigma_2(\mathbf{x})) - g(\mathbf{x})]\, dz_2,$$

where the partial derivative of $g(\mathbf{x})$ with respect to x_i is denoted by $g_{x_i}(\mathbf{x})$.

By the argument made in the main text, finding a new technology implies a reduction in assets. Hence, the Poisson process responsible for reducing the number of assets a is the same as the one increasing the vintage γ. Therefore we set $dz_1 = dz_2 = dq$. Adapting the differential accordingly, we obtain

$$dg(\mathbf{x}) = [g_{x_1}(\mathbf{x})\, f_1(\mathbf{x}) + g_{x_2}(\mathbf{x})\, f_2(\mathbf{x})]\, dt \\ + [g(x_1 + \sigma_1(\mathbf{x}), x_2 + \sigma_2(\mathbf{x})) - g(\mathbf{x})]\, dq. \tag{20}$$

Households accumulate wealth $a = kv$. Given the technologies described in the text, k changes in a deterministic way as a function of savings (expressed in terms of the price v of the capital good) $v^{-1}[w + k\pi - i - e]$, where w is labour income, π are dividend payments per unit of capital, i is expenditure on R & D and e is consumption expenditure. The household's stock of capital k changes in a stochastic way as a function of the outcome of R & D which, if successful, changes k by $iI^{-1}\tilde{K} - s_k k$, which is the household analogue to (4) where the sharing rule has been taken into consideration. Hence, the household's stock of capital evolves according to

$$dk = v^{-1}[w + k\pi - i - e]\, dt + [iI^{-1}\tilde{K} - s_k k]\, dq.$$

The value of a unit of capital v evolves according to

$$dv = \alpha_d v\, dt + \alpha_s v\, dq,$$

where the coefficients a_d and a_s have to be determined in general equilibrium and, therefore, are exogenous to the household.

With the above version (20) of Ito's Lemma and after replacing variables according to $\mathbf{x} = (k, v)$, $f_1(\mathbf{x}) = v^{-1}[w + k\pi - i - e]$, $f_2(\mathbf{x}) = \alpha_d v$, $g(x) = kv$, $\sigma_1(\mathbf{x}) = iI^{-1}\tilde{K} - s_k k$ and $\sigma_2(\mathbf{x}) = \alpha_s v$, the household budget constraint reads

$$\begin{aligned} da &= (vv^{-1}[w + k\pi - i - e] + k\alpha_d v)\, dt + [(k + iI^{-1}\tilde{K} - s_k k)(v + \alpha_s v) - a]\, dq \\ &= (ra + w - i - e)\, dt + (iI^{-1}\varpi - sa)\, dq, \end{aligned}$$

where the interest rate is defined as

$$r \equiv \alpha_d + \frac{\pi}{v}$$

and the "dq-term" follows from simple rearrangements and redefinitions,

$$(k+iI^{-1}\tilde{K}-s_k k)(v+\alpha_s v)-a=iI^{-1}\tilde{K}v(1+\alpha_s)+(1-s_k)\,kv(1+\alpha_s)-a$$
$$\equiv iI^{-1}\varpi-sa,$$

where $s=1-(1-s_k)(1+\alpha_s)$ and $\varpi=\tilde{K}v(1+\alpha_s)$. This shows that stochastic changes in wealth stem both from a quantity channel, i.e. changes in the number of capital units k, and from a price channel, i.e. a change in the value v per unit of capital.

Note that $0\leqslant s_k\leqslant 1$ (the reduction in the physical capital stock can at maximum be 100%) and $\alpha_s\geqslant -1$ (the value of a unit of capital is at least zero) implies $s\leqslant 1$.

APPENDIX 2

The Bellman Equation (7)

Let the notation be as in Appendix 1. According to Kushner [10, p. 20], applying the differential generator $\mathscr{D}$ to $g(\mathbf{x})$ gives

$$\mathscr{D}g(\mathbf{x})=g_{x_1}(\mathbf{x})\,f_1(\mathbf{x})+g_{x_2}(\mathbf{x})\,f_2(\mathbf{x})$$
$$+[\,g(x_1+\sigma_1(\mathbf{x}),x_2)-g(\mathbf{x})]\,a_1+[\,g(x_1,x_2+\sigma_2(\mathbf{x}))-g(\mathbf{x})]\,a_2,$$

where the probability per unit of time dt that x_i jumps with an amplitude of $\sigma_i(\mathbf{x})$ is given by $a_i\,dt$. $\mathscr{D}g(\mathbf{x})$ gives the expected change per unit of time of the function $g(\mathbf{x})$.

Again, with $dz_1=dz_2=dq$, we can adapt Kushner's differential generator to

$$\mathscr{D}g(\mathbf{x})=g_{x_1}(\mathbf{x})\,f_1(\mathbf{x})+g_{x_2}(\mathbf{x})\,f_2(\mathbf{x})+[\,g(x_1+\sigma_1(\mathbf{x}),x_2+\sigma_2(\mathbf{x}))-g(\mathbf{x})]\,a_1.$$

The probability that a jump in $\mathbf{x}$ occurs is given by $a_1\,dt$. If it occurs, $\mathbf{x}$ changes by the amount $\sigma(\mathbf{x})$. Replace variables according to $\mathbf{x}=(a,\gamma)$, $f_1(\mathbf{x})=ra+w-i-e$, $f_2(x)=0$ (as $d\gamma=dq$), $g(\mathbf{x})=V(\mathbf{x})$, $\sigma_1(\mathbf{x})=\varpi(i/I)-sa$, $\sigma_2(\mathbf{x})=1$ and a_1 by the arrival rate λ. The expected change per unit of time dt is then

$$(dt)^{-1}\,\boldsymbol{\varepsilon}\,dV(a)=V_a(a,\gamma)[\,ra+w-i-e\,]$$
$$+\lambda(I)\left[\,V\left(\varpi\frac{i}{I}+(1-s)\,a,\gamma+1\right)-V(a,\gamma)\right].$$

Since the Bellman equation for this problem reads (cf. Dixit and Pindyck [2, ch. 4]) $\rho V(a,\gamma) = \max_{e,\theta}\{u(c) + (1/dt)\, \varepsilon\, dV(a,\gamma)\}$, the household's maximization problem can then be written as in (7).

APPENDIX 3

Deriving Equation (10)

Consider the maximized Bellman equation, where control variables in the Bellman equation (7) were replaced by functions of the state variable a which result from first order conditions,

$$\rho V(a) = u(c(a)) + V_a(a)[ra + w - i(a) - e(a)] + \lambda[V(\tilde{a}(a)) - V(a)]. \tag{21}$$

Asset holdings after successful innovation, $\tilde{a}(a) = \varpi(i(a)/I) + (1-s)a$, depend on current wealth a as current investment i is a function of current wealth. Aggregate investment I is not considered to be a function of current asset holdings a as each household is small. This is in accordance with assumptions made in deriving first order conditions (8) and (9). The differential of (21) with respect to a is[9]

$$\begin{aligned}\rho V_a\, da = {} & u'(c(a))\, dc + V_{aa}[ra + w - i(a) - e(a)]\, da \\ & + V_a[r\, da - di(a) - de(a)] \\ & + \lambda[V_{\tilde{a}}(\tilde{a}(a))[\varpi I^{-1}\, di(a) + (1-s)\, da] - V_a\, da].\end{aligned} \tag{22}$$

As the consumption good price p is an aggregate variable that does not depend on a, we have $de(a) = p\, dc(a)$. Multiplying the first order condition for consumption (8) by de therefore gives $u'(c)\, dc(a) = V_a\, de(a)$ which simplifies (22) to

$$\begin{aligned}\rho V_a\, da = {} & V_{aa}[ra + w - i(a) - e(a)]\, da + V_a[r\, da - di(a)] \\ & + \lambda[V_{\tilde{a}}(\tilde{a}(a))[\varpi I^{-1}\, di(a) + (1-s)\, da] - V_a\, da].\end{aligned} \tag{23}$$

As $(d\tilde{a}/di) = \varpi I^{-1}$, the first order conditions (9) reads

$$V_a(a) = \lambda V_{\tilde{a}}(\tilde{a})\, \frac{\varpi}{I}. \tag{24}$$

[9] In this and the following appendices, dx is used as a short form for $x_a\, da$ and V_a and V_{aa} are used as short forms for $V_a(a)$ and $V_{aa}(a)$, respectively, when no ambiguity arises.

Multiplying by $di(a)$ and inserting simplifies this differential further to[10]

$$\rho V_a\, da = V_{aa}[ra + w - i(a) - e(a)]\, da + V_a r\, da$$
$$+ \lambda[V_{\tilde{a}}(\tilde{a}(a))(1-s)\, da - V_a\, da].$$

By deleting all the das and rearranging we have

$$(\rho - r + \lambda)\, V_a - (1-s)\, \lambda V_{\tilde{a}}(\tilde{a}(a)) = V_{aa}[ra + w - i(a) - e(a)].$$

APPENDIX 4

The Keynes–Ramsey Rule (11)

The differential of the marginal value of a unit of wealth reads

$$dV_a = V_{aa}\, da = V_{aa}[ra + w - i - e]\, dt + V_{aa}\left[\varpi \frac{i}{I} - sa\right] dq \qquad (25)$$

where the last equality follows from inserting the budget constraint (5).

Multiplying (10) by dt and adding $V_{aa}[\varpi(i/I) - sa]\, dq$ gives

$$[\rho - r + \lambda]\, V_a\, dt - [1-s]\, \lambda V_{\tilde{a}}\, dt + V_{aa}\left[\varpi \frac{i}{I} - sa\right] dq$$
$$= V_{aa}[ra + w - i - e]\, dt + V_{aa}\left[\varpi \frac{i}{I} - sa\right] dq = dV_a,$$

where the last equality follows from (25).

Since $u'(c)\, p^{-1} = V_a(a)$, $dV_a = u''(c)\, p^{-1}\, dc - u' p^{-2}\, dp \Leftrightarrow u''(c)\, p^{-1}\, dc = dV_a + (u'/p)(dp/p)$. Further, $V_{aa} = u''(c)\, c_a\, p^{-1}$. Therefore, we have

$$\frac{u''(c)}{p}\, dc = [\rho - r + \lambda]\, V_a\, dt - [1-s]\, \lambda V_{\tilde{a}}\, dt$$
$$+ V_{aa}\left[\varpi \frac{i}{I} - sa\right] dq + \frac{u'}{p} \frac{dp}{p}$$

[10] Clearly, inserting first order conditions simply proves an application of the envelope theorem to the maximization problem at hand.

$$\Leftrightarrow \frac{u''(c)}{p} dc = \left[\rho - r + \frac{dp/dt}{p} + \lambda \right] \frac{u'(c)}{p} dt - [1-s]\, \lambda \frac{u_c(c(\tilde{a}))}{\tilde{p}} dt$$

$$+ u''(c)\, c_a\, p^{-1} \left[\varpi \frac{i}{I} - sa \right] dq$$

$$\Leftrightarrow -\frac{u''(c)}{u'(c)} dc = \left[r - \frac{dp/dt}{p} - \lambda - \rho \right] dt + [1-s]\, \lambda \frac{u'(c(\tilde{a}))\, p}{u'(c(a))\, \tilde{p}} dt$$

$$- \frac{u''(c)\, c_a}{u'(c)} \left[\varpi \frac{i}{I} - sa \right] dq.$$

APPENDIX 5

Proof of Lemma

The derivative (12) implies bang-bang behaviour, of individual investment,

$$i = \begin{Bmatrix} \underline{i} \\ i^* \\ \bar{i} \end{Bmatrix} \Leftrightarrow b \frac{\varpi}{p_I} V_{\tilde{a}}(\varpi m^{-1} + (1-s)a) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} V_a(a),$$

where $\underline{i}$ and $\bar{i}$ are lower and upper bounds (which might be minus and plus infinity). Individual investment behaviour exclusively depends on aggregate variables, and on an individual's stock of wealth.

In the general equilibrium perspective adopted here, the lower investment level is given by

$$\underline{i} = 0.$$

Since each household behaves the same and since it *technologically* unfeasible to allocate a negative amount of resources to R & D, minimum individual investment in the risky asset is zero.

The upper investment level is given by instantaneous savings,

$$\bar{i} = ra + w - e.$$

Assume it lies above this level. Then, once households have passed the threshold level a^* where they start investing in R & D[11], the stock of wealth of the household would decrease and households would again invest in asset accumulation. After some time, they would again invest in R & D

[11] An assumption required here is that the economy starts with a stock of assets such that $a < a^*$.

and so on. As households change instantaneously from 100% investment in asset accumulation to 100% investment in R & D, the aggregate stock of wealth remains unchanged at the threshold level. This means households invest instantaneous savings[12].

Finally, as households switch from $\underline{i}$ to $\bar{i}$ when (12) holds with equality, it follows that

$$i^* = ra + w - e.$$

This implies that the economy will never reach a situation where the returns to investment in the risky asset are strictly larger than returns to investing in the riskless asset,

$$b\frac{\varpi}{p_I} V_{\tilde{a}}(\varpi m^{-1} + (1-s)a) > V_a(a),$$

which completes the proof of the lemma.

APPENDIX 6

Proof of Theorem

The proof has three parts.

(a) "$i = ra + w - e \Rightarrow r - \rho - \lambda[1 - [1-s]\Omega] = 0$"
Consider the differential of the maximized Bellman equation (21) with respect to a, insert the consumption first order condition, delete the das and find

$$\rho V_a = V_{aa}[ra + w - i - e] + V_a[r - i_a] + \lambda[V_{\tilde{a}}[1 - s + \varpi I^{-1} i_a] - V_a].$$

(Note the close similarity to (23) which was derived in a very similar way.) Rearranging yields

$$(\rho + \lambda - r)\, V_a - \lambda[1-s]\, V_{\tilde{a}} = V_{aa}[ra + w - i - e] + i_a\left[\lambda\frac{\varpi}{I} V_{\tilde{a}} - V_a\right]. \quad (26)$$

[12] As suggested by an associate editor, one could allow for an exponential depreciation rate for capital in (1). This would imply that the upper investment level for households would not equal total savings but total savings reduced by a certain amount such that this amount exactly compensates the depreciation rate and the economy's aggregate capital stock remains constant when R & D is undertaken.

An investment level of $i = ra + w - e$ implies together with the lemma (13) that the right-hand side (RHS) of (26) is zero. Replacing the derivatives of the value functions by the expression from the consumption first order condition (8) on the LHS, we find (a). ▮

(b) "$i = 0 \Rightarrow r - \rho - \lambda[1 - [1 - s]\Omega] > 0$"

By (13), $i = 0$ implies that the RHS of (26) is negative as $V_{aa} < 0$, $i_a \geqslant 0$ and $ra + w - e > 0$. Hence, from the LHS, $\rho + \lambda - r - \lambda[1 - s](V_{\tilde{a}}/V_a) < 0$. Inserting (8) and rearranging proves (b). ▮

(c) "(a) and (b) prove the Theorem"

As $r - \rho - \lambda[1 - [1 - s]\Omega]$ can be either zero or positive and as investment can be either zero or equal to savings, (a) implies $r - \rho - \lambda[1 - [1 - s]\Omega] > 0 \Rightarrow i = 0$. This follows from $(x \Rightarrow y) \Rightarrow (\neg y \Rightarrow \neg x)$. By identical reasoning, (b) implies $r - \rho - \lambda[1 - [1 - s]\Omega] = 0 \Rightarrow i = ra + w - e$. ▮

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