

Matching and Saving in Continuous Time: Stability

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- work in progress (proofs complete, writing incomplete) -

We analyse stability properties of a matching and saving model in continuous time. Labour income is stochastic due to random transitions between employment and unemployment. Precautionary saving implies an optimal consumption function and two stochastic processes for wealth and labour income. Existence, uniqueness and stability of the stationary distribution of the wealth-employment process is proven.

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1 Introduction

The analysis of a *deterministic* dynamic system is usually undertaken in three steps: First, one assumes that there is a solution to the system where variables do not change – a steady state. One can then try to prove its existence. Second, the uniqueness of this steady state can be proven. Finally, out-of steady state dynamics are analysed. The central question in this final step is to understand whether transitional dynamics for any reasonable initial condition imply that all variables converge to their steady state levels.

The analysis of *stochastic* dynamic systems follows the same steps. The first step consists in proving existence of a stationary distribution. A stationary distribution is the concept in stochastic systems which corresponds to a steady state in a deterministic system. The second step proves uniqueness of this stationary distribution - similar to proving uniqueness of a steady state. The final question then deals with transitional dynamics - now at the level of distributions. One would like to understand whether (economically reasonable) initial distributions converge to the stationary distribution.

The present paper will undertake these steps for a prototype model of matching and saving in continuous time. Bayer and Wälde (2010a) have extended the textbook matching model of Pissarides (1985, 2000) to allow for savings. Consumption thereby no longer equals current income and consumption smoothing is possible. We call

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the resulting model a prototype model as all features not essential for understanding the impact of savings were removed (in particular, there is competitive factor pricing and there are no vacancies). Equilibrium properties differ dramatically from a model without savings.²

This paper is related to a long history of stochastic dynamic models in economics. - to be written (discrete time models) -

As is true for the derivation of the Fokker-Planck equations in Bayer and Wälde (2010b), the relevance of the proofs provided in this paper go beyond the specific matching and saving model used as background. As we work in continuous time, our analysis is related to the huge literature in economics on continuous time models with uncertainty. The principles of the analysis we undertake can be undertaken for all other continuous time models under uncertainty as well. As – to the best of our knowledge – this has not been undertaken in the economics literature so far, we hope to provide tools here which provide useful in other contexts as well. - to be written (continuous time models) -

The proof of existence and uniqueness of an invariant distribution and of ergodicity, i.e. of convergence to the said distribution, builds on the work of Meyn and Tweedie (1993a,b,c) and Down et al. (1995). In particular, it is worth noting that the wealth process is not smoothing, in the sense that the strong Feller property does not hold. However, in the low-interest rate regime, we can still show a strong version of recurrence (namely Harris recurrence) by using a weaker smoothing property, and thus obtain uniqueness of the invariant distribution. Ergodicity is then implied by a Lyapunov-type condition.

The existence proof for a stationary distribution will be undertaken in sect. 4. The second step in sect. 4 proves uniqueness of this stationary distribution. The final sect. 5 proves that all initial distributions converge to this stationary distribution. Before undertaking these steps, some background is provided in sect. 3. The next section provides the modelling background for the proofs.

2 The model background

The proofs provided in this paper are of interest not only for the specific model by which they are motivated (Bayer and Wälde, 2010a) but also for other stochastic continuous time models. For this reason, we present here only the properties of our model which are essential for the proofs. Applications to other models then simply requires to identify the corresponding properties of other models and adapt the proofs presented below.

There are two variables describing the state of an individual. They are wealth $a(\tau)$ and the labour income $z(\tau)$ of this individual. The individual behaves optimally

²A companion paper (Bayer and Wälde, 2010b) proves central equilibrium properties of the matching and saving model. Conditions for rising consumption are proven and the existence of an optimal consumption path is shown. Going in its relevance beyond the matching and saving model, Fokker-Planck equations are derived. These equations (partial and ordinary differential equation systems) describe the density of the wealth-employment process as predicted by the model.

according to some optimality criterion which implies a consumption function $c(\tau) = c(a(\tau), z(\tau))$. Wealth satisfies the stochastic differential equation

$$da(\tau) = (ra(\tau) + z(\tau) - c(\tau))d\tau, \quad (1)$$

where r denotes the constant interest rate. Labour income is modeled by a continuous-time Markov chain with two possible states w and b . Here, $w > b$ is assumed to be the net salary from employment, while we think of b as an unemployment benefit. The transition rate from state b to state w is the “job arrival rate” $\mu > 0$, the transition rate from w to b is the “separation rate” $s > 0$.

While $c(a(\tau), z(\tau))$ is defined as the utility maximizing consumption function, it was analytically characterized by the solution of a system of ordinary differential equations (with singularity - see equations (20) in Bayer and Wälde, 2010a) on a domain $[-b/r, a_w^*] \times \{w, b\}$. Now consider the following assumptions: (i) $r < \rho$, where ρ is the time preference rate of the optimising individual, (ii) $c(a, z)$ is continuously differentiable in a on $[-b/r, a_w^*] \times \{w, b\}$, (iii) relative consumption $c(a, w)/c(a, b)$ is continuously differentiable in a and the derivative changes its sign only finitely often in every finite interval and (iv) the initial wealth $a(t)$ is chosen inside the interval $[-b/r, a_w^*]$. Under these assumptions, it has been revealed that wealth $a(\tau)$ is increasing in time when $z(\tau) = w$ and decreasing when $z(\tau) = b$ (strictly so when $a(\tau) \in] -b/r, a_w^*]$). Moreover, wealth will never leave the interval $[-b/r, a_w^*]$.³

This presentation of essential features of our model for the stability analysis has shown that the objective function hardly plays any role. As long as there is a consumption function $c(\cdot)$ with properties as presented above, the proofs go through also for other setups. Similarly for the state variables. As long as the properties of the state variables in other models are the same (as an example: wealth is often bounded in general equilibrium models of precautionary saving), the proofs here can directly be applied. The same holds true for factor pricing or the competitive structure. This would affect the levels of labour income which z can take or the shape of $c(\cdot)$. But it would not affect the validity of the proofs. If properties differ slightly, the proofs below will show how they need to be adjusted.

The goal of the present paper is to prove theorem 2 in Bayer and Wälde (2010a), i.e. we prove that the system in the low interest rate case $r < \rho$ has a unique invariant distribution, and, moreover, that the system is ergodic in the sense that the distribution always converges to the invariant distribution as $\tau \rightarrow \infty$. To this end, we are first going to review some important results from ergodic theory of Markov processes. As references, we mainly use the works of Meyn and Tweedie (1993b,c), see also their book (1993a) about Markov processes in *discrete* time.

³The domain where economically relevant equilibrium dynamics takes place is $[-b/r, a_w^*]$. If wealth is larger than a_w^* , it will always decrease, regardless of z . It has been proven that wealth levels smaller than $-b/r$ can not occur in equilibrium as $-b/r$ forms a natural borrowing constraint.

3 Review of ergodicity results for Markov processes

Let $(X_t)_{t \geq 0}$ be a (homogeneous) Markov process with the state space \mathbf{X} , where \mathbf{X} is assumed to be locally compact and separable metric space, which is endowed with its Borel σ -algebra $\mathcal{B}(\mathbf{X})$.⁴ Let $P^t(x, A)$, $t \geq 0$, $x \in \mathbf{X}$, $A \in \mathcal{B}(\mathbf{X})$, denote the corresponding transition kernel, i.e.

$$P^t(x, A) = P(X_t \in A | X_0 = x) = P_x(X_t \in A),$$

where P_x is a shorthand-notation for the conditional probability $P(\cdot | X_0 = x)$. Note that $P^t(\cdot, \cdot)$ is a *Markov kernel*, i.e. for every $x \in \mathbf{X}$, the map $A \mapsto P^t(x, A)$ is a probability measure on $\mathcal{B}(\mathbf{X})$ and for every $A \in \mathcal{B}(\mathbf{X})$, the map $x \mapsto P^t(x, A)$ is a measurable function. Similarly, by a *kernel* we understand a function $K : (\mathbf{X}, \mathcal{B}(\mathbf{X})) \rightarrow \mathbb{R}_{\geq 0}$ such that $K(x, \cdot)$ is a measure, not necessarily normed by 1, for every x and $K(\cdot, A)$ is a measurable function for every measurable set A . Moreover, let us denote the corresponding semi-group by P_t , i.e.

$$P_t f(x) = E(f(X_t) | X_0 = x) = \int_{\mathbf{X}} f(y) P^t(x, dy)$$

for $f : \mathbf{X} \rightarrow \mathbb{R}$ bounded measurable. Before adapting two of the most fundamental notions of the classical theory of Markov chains to the continuous case, let us introduce two relevant random variables. For a measurable set A , we consider

$$\tau_A = \inf\{t \geq 0 | X_t \in A\}, \quad \eta_A = \int_0^\infty \mathbf{1}_A(X_t) dt.$$

Definition 3.1 *Assume that there is a σ -finite, non-trivial measure φ on $\mathcal{B}(\mathbf{X})$ such that, for sets $B \in \mathcal{B}(\mathbf{X})$, $\varphi(B) > 0$ implies $E_x(\eta_B) > 0$, $\forall x \in \mathbf{X}$. Here, similar to P_x , E_x is a short-hand notation for the conditional expectation $E(\cdot | X_0 = x)$. Then X is called φ -irreducible.*

If X is a φ -irreducible process for some irreducibility measure φ , then there is a *maximal irreducibility measure* ψ , which is an irreducibility-measure, such that every other irreducibility measure φ is absolutely continuous with respect to ψ .⁵ We will often simply refer to irreducible processes, without mentioning an irreducibility measure.

Definition 3.2 *The process X is called Harris recurrent if there is a non-trivial σ -finite measure φ such that $\varphi(A) > 0$ implies that $P_x(\eta_A = \infty) = 1$, $\forall x \in \mathbf{X}$. Moreover, if a Harris recurrent process X has an invariant probability measure, then it is called positive Harris.*

⁴The state space of the employment-wealth process $\mathbf{X} = \mathbb{R} \times \{w, b\}$, endowed with the usual Euclidean metric in the first component, the discrete metric in the second component, and the usual product metric on \mathbf{X} obviously satisfies these requirements. We will apply the results of this section to the Markov process $X_\tau = (a(\tau), z(\tau))$.

⁵Given two measures μ and ν , μ is called absolutely continuous with respect to ν , if $\nu(A) = 0$ implies $\mu(A) = 0$, for every measurable set A . In that case, there exists a density of μ with respect to ν .

Remark 3.3 *Intuitively, a Markov process is irreducible, if every state can be reached from every other state with positive probability. Definition 3.1 gives the appropriate generalization of this concept to the continuous situation.*

On the other hand, Harris recurrence means that every state – i.e. every set of positive φ -measure – is visited infinitely often. In particular, Harris recurrence implies irreducibility (with irreducibility-measure φ). Notice that there is also the weaker notion of recurrence.

Remark 3.4 *Harris recurrence may be equivalently defined by the existence of a σ -finite measure μ such that $\mu(A) > 0$ implies that $P_x(\tau_A < \infty) = 1$.*

The following proposition is cited in Meyn and Tweedie (1993b, page 491). It is the analogue to a well-known statement for Markov chains.

Proposition 3.5 *If the Markov process X is Harris recurrent and φ -irreducible, then there is a unique invariant measure.*

A simple sufficient condition for irreducibility is given in Meyn and Tweedie (1993b, prop. 2.1).

Proposition 3.6 *Suppose that there exists a σ -finite measure μ such that $\mu(B) > 0$ implies that $P_x(\tau_B < \infty) > 0$. Then X is φ -irreducible, where*

$$\varphi(A) \equiv \int_{\mathbf{X}} R(x, A) \mu(dx), \quad R(x, A) \equiv \int_0^\infty P^t(x, A) e^{-t} dt.$$

Definition 3.7 *Given a probability measure a on $\mathbb{R}_{\geq 0}$, define a Markov kernel K_a by*

$$K_a(x, A) \equiv \int_0^\infty P^t(x, A) a(dt).$$

The Markov chain (in discrete time and with state space \mathbf{X}) with transition kernel K_a is called K_a -chain.

K_a -chains are important tools, because they often have the same properties as the underlying, continuous-time Markov-process. In particular, if any K_a -chain is Harris recurrent, then so is the Markov process X , cf. Meyn and Tweedie (1993b, th. 2.2).

It is well known that existence of an invariant measure is intimately connected with growth properties of the Markov process. We will formulate three such properties, which we will need below.

Definition 3.8 *We say that $X \rightarrow \infty$ if $X_t \in C^c$ for every compact set C and every t large enough. The Markov process X is called non-evanescent if $P_x(X \rightarrow \infty) = 0$ for every $x \in \mathbf{X}$.*

Definition 3.9 *A family of probability measures μ_λ , $\lambda \in \Lambda$, is called tight if for every $\epsilon > 0$ there is a compact set C such that*

$$\mu_\lambda(C) \geq 1 - \epsilon, \quad \forall \lambda \in \Lambda.$$

Remark 3.10 *By a classical result by Prokhorov, every tight family of random variables has a sub-sequence, which converges in distribution.*

We will not use the notion of tightness, but instead a slight generalization, which turns out to be very useful for our purposes.

Definition 3.11 *The process X is called bounded in probability on average if for every $x \in \mathbf{X}$ and every $\epsilon > 0$ there is a compact set $C \subset \mathbf{X}$ such that*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t P_x(X_s \in C) ds \geq 1 - \epsilon,$$

for every starting value $x \in \mathbf{X}$.

Finally, let us recall the following important continuity notions.

Definition 3.12 *The Markov process X satisfies the weak Feller condition if for every continuous bounded function $f : \mathbf{X} \rightarrow \mathbb{R}$ the function $P_t f : \mathbf{X} \rightarrow \mathbb{R}$ is again continuous. Moreover, if $P_t f$ is continuous even for every bounded measurable function f , then X satisfies the strong Feller condition.*

Remark 3.13 *The employment-wealth process obviously satisfies the weak Feller condition (because the wealth process is path-wise continuous in the starting value), but it does not satisfy the strong Feller condition. An example for a strong Feller process would be the Brownian motion. One should remark that our usage of the Feller conditions is common, but not canonical in the literature.*

Meyn and Tweedie (1993b, th. 3.1) gives a simple condition for the existence of an invariant probability measure.

Proposition 3.14 *If a Markov process X satisfies the weak Feller condition and is bounded in probability on average, then there is an invariant probability measure for X .*

Finally, for a related uniqueness result, we need yet another continuity result, which is conceptually between the notions of weak and strong Feller.

Definition 3.15 *The Markov process X is called T-process, if there is a probability measure a on $\mathbb{R}_{\geq 0}$ and a kernel T on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ satisfying the following three conditions:*

1. *For every $A \in \mathcal{B}(\mathbf{X})$, the function $x \mapsto T(x, A)$ is lower semi-continuous⁶.*

⁶A function $f : \mathbf{X} \rightarrow \mathbb{R}$ is called lower semi-continuous if

$$\forall x_0 \in \mathbf{X} : \liminf_{x \rightarrow x_0} f(x) \geq f(x_0).$$

In particular, every continuous function is lower-semicontinuous.

2. For every $x \in \mathbf{X}$ and every $A \in \mathcal{B}(\mathbf{X})$ we have $K_a(x, A) \geq T(x, A)$.
3. $T(x, \mathbf{X}) > 0$ for every $x \in \mathbf{X}$.

The uniqueness result follows from Meyn and Tweedie (1993b, th. 3.2), which we report below as our proposition 3.16, and proposition 3.5.

Proposition 3.16 *Suppose that X is a φ -irreducible T -process. It is bounded in probability on average if and only if it is Harris recurrent.*

4 Existence and uniqueness of invariant measures for the employment-wealth process

We concentrate on the low-interest-rate regime $r < \rho$. Then we know that the support of the law of $a(\tau)$ is a subset of $[-b/r, a_w^*]$. Therefore, the distribution of $(a(\tau), z(\tau))$ understood as a family of probability measures is tight in the sense of definition 3.9. Since the process satisfies the weak Feller property, proposition 3.14 implies the existence of an invariant probability measure. In order to show uniqueness, we need to prove that $(a(\tau), z(\tau))$ is Harris recurrent. Formally, we choose $\mathbf{X} = [-b/r, a_w^*] \times \{w, b\}$.

Lemma 4.1 *In the low-interest-regime with $r < \rho$, $(a(\tau), z(\tau))$ is an irreducible Markov process.*

Proof. Let $-b/r < a < a_w^*$, $z \in \{w, b\}$. Then, regardless of the initial point $a_t \in [-b/r, a_w^*]$ and regardless of z_t , it is possible to attain the state (a, z) in finite time with probability greater than zero. Thus, proposition 3.6 implies irreducibility. ■

Remark 4.2 *In the case of intermediate interest rate, the process is not irreducible. Indeed, if the wealth surpasses the value a_b^* , then it increases forever. Therefore, it is impossible to return to a state with lower wealth. The same holds true for the high-interest-rate-regime.*

We continue with the following useful auxiliary lemma.

Lemma 4.3 *The conditional density of the time of the first jump in employment given that there is precisely one such jump in $[0, \tau]$ and that $z(0) = w$ is given by*

$$g_\tau^{(1)}(u) = \frac{\mu - s}{e^{(\mu-s)\tau} - 1} e^{(\mu-s)u}, \quad 0 \leq u \leq \tau.^7$$

⁷In the case $\mu = s$ the formula is still correct in the sense of a limit $\lim_{\mu \rightarrow s}$. Since all the derivations below equally go through in that case, we will tacitly omit the distinction between the cases $\mu \neq s$ and $\mu = s$.

Proof. The joint probability of the first jump $\tau_1 \leq u \leq \tau$ and $N_\tau = 1$, where N_τ denotes the number of jumps in $[0, \tau]$, is given by

$$\begin{aligned} P(\tau_1 \leq u, N_\tau = 1) &= P(\tau_1 \leq u, \tau_2 \geq \tau - \tau_1) \\ &= \int_0^u P(\tau_2 \geq \tau - v) s e^{-sv} dv \\ &= \int_0^u e^{-\mu(\tau-v)} s e^{-sv} dv \\ &= \frac{s}{\mu - s} e^{-\mu\tau} \left(e^{(\mu-s)u} - 1 \right). \end{aligned}$$

Here, τ_2 denotes the time between the first and the second jump, and we have used independence of τ_1 and τ_2 . Dividing through the probability of $N_\tau = 1$, we get

$$P(\tau_1 \leq u | N_\tau = 1) = \frac{e^{(\mu-s)u} - 1}{e^{(\mu-s)\tau} - 1},$$

and we obtain the above density by differentiating with respect to u . ■

Now we formulate the central technical lemma.

Lemma 4.4 *Assume that the optimal consumption function $c(a, z)$ is C^1 in wealth a . Then the employment-wealth-process $(a(\tau), z(\tau))$ is a T -process.*

Proof. Given $(a_t, z_t) \in \mathbf{X}$ and $A \in \mathcal{B}(\mathbf{X})$, the probability of $(a(\tau), z(\tau)) \in A$ for some $\tau > t$ can naturally be decomposed as

$$\begin{aligned} P((a(\tau), z(\tau)) \in A | a(t) = a_t, z(t) = z_t) &= \\ \sum_{k=0}^{\infty} P((a(\tau), z(\tau)) \in A | a(t) = a_t, z(t) = z_t, N(\tau) = k) \cdot P(N(\tau) = k) & \\ \equiv \sum_{k=0}^{\infty} f_\tau^{(k)}(a_t, z_t; A) P(N(\tau) = k), & \end{aligned}$$

where $N(\tau)$ denotes the number of jumps of the employment state between t and τ . The idea of the argument is that $f^{(0)}$ – corresponding to the event that no jump has occurred – has no smoothing property whatsoever, but all the other $f^{(k)}$, $k > 0$, do have smoothing effects, because of the random time of the jump. Therefore, we are going to prove that $f^{(1)}$ satisfies the conditions of Definition 3.15 on the kernel T (for $a = \delta_\tau$, i.e. $K_a = P^{\tau-t}$).

For ease of notation, let us assume that $z_t = w$ and that $t = 0$ – of course, it is easy to extend the argument to the general case. Let $g_\tau^{(1)}$ denote the density of the time of the first jump in employment conditional on the assumption that there is precisely one such jump in $[0, \tau]$. This density is given by

$$g_\tau^{(1)}(u) = \frac{\mu - s}{e^{(\mu-s)\tau} - 1} e^{(\mu-s)u}, \quad 0 \leq u \leq \tau,$$

see lemma 4.3 above.

Given that the only jump in employment between 0 and τ happens at time $0 \leq u \leq \tau$, let us denote the mapping between the starting value a_t and the value of the wealth at time τ by $\phi_1(a_t, u; \tau)$. i.e. ϕ_1 is the solution map of the stochastic differential equation for the wealth, given that only one jump of z happens in the considered time-interval. By adding the time of that single jump as a variable, ϕ_1 is a deterministic function. We can understand it as the flow generated by the underlying ODE. Indeed, let ψ_z denote the solution map of the ODE

$$\frac{da(\tau)}{d\tau} = ra(\tau) + z - c(a(\tau), z) \quad (2)$$

for $z \in \{w, b\}$ fixed. More precisely, $\psi_z(a, u)$ denotes the solution of the ODE (2) evaluated at time u given that the starting value (at time 0) is a . Then we may obviously write

$$\phi_1(a_t, u; \tau) = \psi_b(\psi_w(a_t, u), \tau - u). \quad (3)$$

Using the function ϕ_1 and the fact that $z(\tau) = b$ (given that $z_t = w$ and that there is precisely one jump in $[0, \tau]$), we may re-write $f_\tau^{(1)}$ as

$$\begin{aligned} f_\tau^{(1)}(a_t, w; A) &= \int_0^\tau \mathbf{1}_A(\phi_1(a_t, u; \tau), b) g_\tau^{(1)}(u) du \quad (4) \\ &= \frac{\mu - s}{e^{(\mu-s)\tau} - 1} \int_0^\tau \mathbf{1}_A(\phi_1(a_t, u; \tau), b) e^{(\mu-s)u} du \\ &= \frac{\mu - s}{e^{(\mu-s)\tau} - 1} \int_{l(a_t, \tau)}^{u(a_t, \tau)} \mathbf{1}_A(y, b) e^{(\mu-s)\phi_1^{-1}(a_t, y; \tau)} \left| \frac{\partial}{\partial y} \phi_1^{-1}(a_t, y; \tau) \right| dy. \quad (5) \end{aligned}$$

In equation (5) we have made the substitution

$$y \equiv \phi_1(a_t, u; \tau),$$

understood as a change from the u -variable to the y -variable. Consequently, $y \mapsto \phi_1^{-1}(a_t, y; \tau)$ is to be understood as the inverse map of $u \mapsto \phi_1(a_t, u; \tau)$, with a_t and τ being hold fix. Therefore, we first need to check the validity of that substitution. Note that $u \mapsto \phi_1(a_t, u; \tau)$ is a strictly increasing map by our assumption. Therefore, the substitution $y = \phi_1(a_t, u; \tau)$ is well-defined. Moreover, this implies that the lower and upper boundaries of the integration in (5) are given by

$$l(a_t, \tau) \equiv \phi_1(a_t, 0; \tau), \quad u(a_t, \tau) \equiv \phi_1(a_t, \tau; \tau),$$

respectively. In order to justify the application of the substitution rule, we need to establish continuous differentiability of ϕ_1^{-1} in y . Formally, we immediately have that

$$\frac{\partial}{\partial y} \phi_1^{-1}(a_t, y; \tau) = \frac{1}{\left. \frac{\partial}{\partial u} \phi_1(a_t, u; \tau) \right|_{u=\phi_1^{-1}(a_t, y; \tau)}}.$$

By (3), we have

$$\begin{aligned}
\frac{\partial}{\partial u} \phi_1(a_t, u; \tau) &= \frac{\partial}{\partial u} \psi_b(\psi_w(a_t, u), \tau - u) \\
&= -\frac{\partial \psi_b}{\partial u}(\psi_w(a_t, u), \tau - u) + \frac{\partial \psi_b}{\partial a}(\psi_w(a_t, u), \tau - u) \frac{\partial \psi_w}{\partial u}(a_t, u) \\
&= -\underbrace{\left[r\psi_b(\psi_w(a_t, u), \tau - u) + b - c(b, \psi_b(\psi_w(a_t, u), \tau - u)) \right]}_{<0} + \\
&\quad + \underbrace{\frac{\partial \psi_b}{\partial a_t}(\psi_w(a_t, u), \tau - u)}_{\geq 0} \underbrace{\left[r\psi_w(a_t, u) + w - c(w, \psi_w(a_t, u)) \right]}_{\geq 0} > 0.
\end{aligned}$$

The inequality follows, because the wealth of the unemployed is, by assumption, strictly decreasing, the final wealth is an increasing function of the initial wealth, and the wealth of the employed is increasing. Moreover, we can easily see that all the terms are continuous in u .

Thus, we can justify the substitution (5). This implies continuity of $f_\tau^{(1)}(a_t, w; A)$ in a_t , since the integrand as well as the integration-boundaries in (5) are continuous in a_t . Therefore, $T(a_t, z_t; A) \equiv f_\tau^{(1)}(a_t, z_t; A)$ satisfies the first condition in Definition 3.15 – continuity in the z -variable is trivial. The second and the third conditions are obvious, and we have finished the proof of the lemma. ■

Theorem 4.5 *Suppose that $r < \rho$ and that $c(a, z)$ is continuously differentiable in a . Then there is a unique invariant probability measure for the employment-wealth process $(a(\tau), z(\tau))$.*

Proof. By lemma 4.1 and lemma 4.4, the employment-wealth process $(a(\tau), z(\tau))$ is an irreducible T -process. Moreover, since it is compactly supported (in fact, the state space \mathbf{X} itself is compact), it is obviously bounded in probability on average. Thus, proposition 3.16 implies that $(a(\tau), z(\tau))$ is Harris recurrent. By proposition 3.5, there is a unique invariant measure (up to a constant multiplier), and proposition 3.14, finally, implies that we may choose the invariant measure to be a probability measure. ■

5 Ergodicity of the employment-wealth process

In this section, we are going to show that the employment-wealth process is ergodic. In fact, it will turn out to be a simple consequence of the T -property, already established in lemma 4.4, and the ergodicity results for general Markov-processes as presented in Down, Meyn and Tweedie (1995).

Definition 5.1 *Given a probability measure a on $\mathbb{R}_{\geq 0}$ and a non-trivial measure ν_a on $\mathcal{B}(\mathbf{X})$. A set $C \in \mathcal{B}(\mathbf{X})$ is called ν_a -petite, if for all $x \in C$ and all $A \in \mathcal{B}(\mathbf{X})$ we have $K_a(x, A) \geq \nu_a(A)$. We call C petite, if the measure ν_a does not matter.*

Let us first recall that every compact set is petite for a non-evanescent, irreducible T -process, see Theorem 4.1 Meyn and Tweedie (1993b, th. 4.1). Since we have already showed that the employment-wealth process satisfies these assumptions for $r < \rho$ (Lemma 4.1 and Lemma 4.4), it immediately follows that every compact set is petite for the employment-wealth process.

Definition 5.2 *Given a measurable function $V \geq 1$ defined on the state space \mathbf{X} of a Markov-process X_t with transition kernel P^t and the unique invariant measure π , we call the Markov-process V -uniformly ergodic, if we can find constants $D < \infty$ and $0 \leq \omega < 1$ such that*

$$\|P^t(x, \cdot) - \pi\|_V \leq V(x)D\omega^t \quad (6)$$

for every $x \in \mathbf{X}$ and $t \geq 0$. Here, $\|\cdot\|_V$ is a generalization of the well known total variation norm defined by

$$\|\lambda\|_V = \sup_{|f| \leq V} \left| \int f(y)\lambda(dy) \right|$$

for a probability measure λ . (The total variation norm is given by $\|\cdot\|_1$, i.e. with $V \equiv 1$.)

Next we present a sufficient condition for V -uniform ergodicity given in Down, Meyn and Tweedie (1995, th. 5.2).

Proposition 5.3 *Given an irreducible, aperiodic Markov-process X_t with infinitesimal generator \mathcal{A} . Assume we can find a measurable function $V \in \mathcal{D}(\mathcal{A})$ such that $V \geq 1$, constants $d, c > 0$ and a petite set C satisfying*

$$\mathcal{A}V \leq -cV + d\mathbf{1}_C.$$

Then the Markov-process is V -uniformly ergodic.

Here, a ψ -irreducible Markov process X_t as before is called *aperiodic*, if there is a petite set C with $\psi(C) > 0$ and a time $T > 0$ such that $P^t(x, C) > 0$ for all $x \in C$ and $t \geq T$, see Down, Meyn and Tweedie (1995, page 1675). For a more comprehensive view on the notion of aperiodicity for Markov chains with general state spaces (but in discrete time), see Meyn and Tweedie (1993a, page 119 ff.).

Corollary 5.4 *Under the assumptions of theorem 4.5, the employment-wealth process is uniformly ergodic.*

Proof. Recall that the employment-wealth-process is a T -process, see lemma 4.4. Therefore, every compact set is petite. Moreover, we have shown irreducibility in lemma 4.1. Without loss of generality, we may restrict the state space to the compact set $[-b/r, a_w^*] \times \{w, b\}$. Consequently, aperiodicity in the sense of the above definition is trivial.

Thus, we are now in the framework of proposition 5.3. Choose $V : [-b/r, a_w^*] \times \{w, b\} \rightarrow \mathbb{R}$ by setting $V \equiv 1$. Then $V \in \mathcal{D}(\mathcal{A})$ and, clearly,

$$\sup_{a \in [-b/r, a_w^*], z \in \{w, b\}} \mathcal{A}V(a, z) < \infty.$$

Consequently, the main condition of Proposition 5.3 can be satisfied with $c = 1$, $C = [-b/r, a_w^*] \times \{w, b\}$ and

$$d \geq \sup_{a \in [-b/r, a_w^*], z \in \{w, b\}} \mathcal{A}V(a, z) + 1.$$

■

6 Conclusion

This paper has proven existence, uniqueness and stability of the stationary distribution of the wealth-employment process implied by a prototype model of matching and saving. - to be written -

Section 3 establishes the existence and uniqueness of an invariant distribution and shows ergodicity, i.e. convergence of the distribution of the process to the invariant distribution. Again, we employ quite general mathematical tools from the theory of Markov processes. We remark that we have to use more advanced methods than in the case of a stochastic differential equation driven by a Brownian motion. Indeed, in that case the strong smoothing properties of Brownian motion can be used to obtain the strong Feller property. Consequently, the corresponding analysis for stochastic models partly driven by a Brownian motion (in the sense that we may also allow for jumps), will often be easier than the one presented in this paper. On the other hand, the analysis of section 3 might be adapted to other stochastic models driven by pure jump processes. In general, it might, however, be non-trivial to establish the conditions of prop. 5.3.

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