Referees' appendix to
The Dynamics of Distributions in Continuous-Time Stochastic Models

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B Consumption and wealth dynamics

This appendix provides preliminary results for deriving the generalized Keynes-Ramsey rules and which allow us to perform the phase diagram illustration and the subsequent existence proof for an optimal consumption path $c(a,z)$.

B.1 Derivation of the Keynes-Ramsey rules (4a) and (4c)

- Bellman equations and first-order conditions

We now let the individual maximize her objective function by choosing a path $\{c(\tau)\}$ of consumption subject to the budget constraint (2) and the equation for its employment status (3). Given that the state variables describing an individual are not only current labour income $z(t)$ but also current wealth $a(t)$, we define the value function as $V(a(t), z(t)) = \max_{\{c(\tau)\}} U(t)$ subject to (3) and (2). The Bellman equation for this problem reads (see Wälde, 2012, part IV)

$$\rho V(a(t), z(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{1}{dt} E_t dV(a(t), z(t)) \right\}. \tag{43}$$

Computing the differential $dV(a(t), z(t))$, taking the constraints (2) and (3) into account and forming expectations yields, suppressing the time argument t for simplicity,

$$\rho V(a,z) = \max_c \left\{ \begin{array}{c} u(c) + [ra + z - c] V_a(a,z) \\ +s(z)[V(a,b) - V(a,w)] + \mu(z)[V(a,w) - V(a,b)] \end{array} \right\}, \tag{44}$$

where V_a stands for the partial derivative of V with respect to a. Note that this Bellman equation holds for both the employment state and the unemployment state as the arrival rates are state dependent.

Given that the individual only needs to choose consumption, the only first-order condition equates marginal utility from consumption with the shadow price of wealth,

$$u'(c(a,z)) = V_a(a,z). \tag{45}$$

We know by the budget constraint (2) that one unit of consumption costs one unit of wealth. Hence, in the optimum, the instantaneous increase in utility due to marginally consuming more is identical to the present value increase in overall utility due to an additional unit of wealth.

- Evolution of the shadow price

Using the budget constraint (2) and the evolution of labour income (3), the differential of the shadow price of wealth reads

$$dV_a(a,z) = V_{aa}(a,z)\{ra+z-c\}\,dt + [V_a(a,w)-V_a(a,b)]\,dq_\mu + [V_a(a,b)-V_a(a,w)]\,dq_s. \tag{46}$$

The maximized version of the Bellman equation (44) simply replaces the control variable c by its optimal value $c(a,z)$,

$$\rho V(a,z) = \left\{ \begin{array}{c} u(c(a,z)) + [ra+z-c(a,z)]\,V_a(a,z) \\ +s(z)[V(a,b)-V(a,w)] + \mu(z)[V(a,w)-V(a,b)] \end{array} \right\}. \tag{47}$$

Differentiating with respect to wealth yields, using the envelope theorem,

$$\rho V_a(a,z) = \left\{ \begin{array}{c} rV_a(a,z) + [ra+z-c(a,z)]\,V_{aa}(a,z) \\ +s(z)[V_a(a,b)-V_a(a,w)] + \mu(z)[V_a(a,w)-V_a(a,b)] \end{array} \right\}. \tag{48}$$

Rearranging yields

$$\begin{aligned} &(\rho-r)V_a(a,z) - s(z)[V_a(a,b)-V_a(a,w)] - \mu(z)[V_a(a,w)-V_a(a,b)] \\ &= [ra+z-c(a,z)]\,V_{aa}(a,z). \end{aligned}$$

Inserting into (46) gives

$$\begin{aligned} dV_a(a,z) = &\{(\rho-r)V_a(a,z) - s(z)[V_a(a,b)-V_a(a,w)] - \mu(z)[V_a(a,w)-V_a(a,b)]\}\,dt \\ &+ [V_a(a,w)-V_a(a,b)]\,dq_\mu + [V_a(a,b)-V_a(a,w)]\,dq_s. \end{aligned}$$

- Inserting first-order condition

When we now replace the shadow price by marginal utility from the first-order condition (45), we get the Keynes-Ramsey rule for marginal utility,

$$\begin{aligned} du'(c(a,z)) = &\left\{ \begin{array}{c} (\rho-r)u'(c(a,z)) - s(z)[u'(c(a,b))-u'(c(a,w))] \\ -\mu(z)[u'(c(a,w))-u'(c(a,b))] \end{array} \right\} dt \\ &+ [u'(c(a,w))-u'(c(a,b))]\,dq_\mu + [u'(c(a,b))-u'(c(a,w))]\,dq_s. \end{aligned} \tag{49}$$

For an employed individual where $\mu(z)=0$ and $a=a_w$, this reads

$$\begin{aligned} du'(c(a_w,w)) = &\{(\rho-r)u'(c(a_w,w)) - s[u'(c(a_w,b))-u'(c(a_w,w))]\}\,dt \\ &+ [u'(c(a_w,b))-u'(c(a_w,w))]\,dq_s. \end{aligned}$$

Let $f(.)$ be the inverse function for u', i.e. $f(u')=c$ and apply the CVF to $f(u'(c(a_w,w)))$ This gives

$$\begin{aligned} df(u'(c(a_w,w))) = &f'(u'(c(a_w,w)))\{(\rho-r)u'(c(a_w,w)) - s[u'(c(a_w,b))-u'(c(a_w,w))]\}\,dt \\ &+ [f(u'(c(a_w,b))) - f(u'(c(a_w,w)))]\,dq_s. \end{aligned}$$

As $f(u')=c$ and therefore $f'(u'(c(a_w,w))) = \frac{df(u'(c(a_w,w)))}{du'(c(a_w,w))} = \frac{dc(a_w,w)}{du'(c(a_w,w))} = \frac{1}{u''(c(a_w,w))}$, we get

$$\begin{aligned} dc(a_w,w) = &\frac{1}{u''(c(a_w,w))}\{(\rho-r)u'(c(a_w,w)) - s[u'(c(a_w,b))-u'(c(a_w,w))]\}\,dt \\ &+ [c(a_w,b)-c(a_w,w)]\,dq_s \Leftrightarrow \\ \frac{u''(c(a_w,w))}{u'(c(a_w,w))}dc(a_w,w) = &\left\{\rho - r - s\left[\frac{u'(c(a_w,b))}{u'(c(a_w,w))} - 1\right]\right\} dt \\ &+ \frac{u''(c(a_w,w))}{u'(c(a_w,w))}[c(a_w,b)-c(a_w,w)]\,dq_s. \end{aligned}$$

Using the instantaneous CRRA utility function (1), we get $\frac{u''(c(a_w,w))}{u'(c(a_w,w))} = \frac{-\sigma c(a_w,w)^{-\sigma-1}}{c(a_w,w)^{-\sigma}} = \frac{-\sigma}{c(a_w,w)}$ and therefore

$$-\sigma \frac{dc(a_w,w)}{c(a_w,w)} = \left\{ \rho - r - s \left[\left(\frac{c(a_w,w)}{c(a_w,b)} \right)^{\sigma} - 1 \right] \right\} dt - \sigma \left[\frac{c(a_w,b)}{c(a_w,w)} - 1 \right] dq_s.$$

After dividing by $-\sigma$, we get (4a) in the main text. The derivation of (4c) also starts from (49) and steps are in perfect analogy.

B.2 Consumption growth and the interest rate

Our analysis focuses on paths $c(a,z)$ as depicted in fig. 1. In this figure, we implicitly considered solutions of our system in the set $Q = \{a \geq -b/r\} \cap \{c(a,w) \leq ra + w\} \cap \{c(a,b) \geq ra + b\} \cap \{c(a,b) \geq 0\} \cap \{c(a,w) \geq c(a,b)\}$. In words, wealth is at least as large as the maximum debt level b/r, consumption of the employed worker is below the zero-motion line for her wealth, consumption of the unemployed worker is above her zero-motion line for wealth, consumption of the unemployed worker is non-negative and consumption of employed workers always exceeds consumption of unemployed workers (see lem. B.12).

For the proofs we restrict this set in two ways. First, we consider the domain

$$Q_v = \{(a, c(a,w), c(a,b)) \in \mathbb{R}^3 | (a, c(a,w), c(a,b)) \in Q, c(a,w) \leq ra + w - v\}, \tag{50}$$

where v is the small positive constant, as an approximation to our "full" set Q. As $Q_0 = Q$, Q_v simply excludes the zero-motion line for wealth of the employed workers. We need to do this as the fraction on the right-hand side of our differential equation (5a) is not defined for the TSS.[20] As v is small, however, we can get arbitrarily close to this zero-motion line and Q_v approximates Q arbitrarily well.

Second, we consider

$$R_{v,\Psi} = \{(a, c(a,w), c(a,b)) \in \mathbb{R}^3 | (a, c(a,w), c(a,b)) \in Q_v, \; c(a,w) \leq \Psi < \infty\}, \tag{51}$$

where Ψ is a finite large constant.[21] This additional restriction makes the set $R_{v,\Psi}$ bounded. This is a purely technical necessity.

We first focus on individuals in periods between jumps. The evolution of consumption is then given by the deterministic part, i.e. the dt-part, in (4a) and (4c). We then easily understand

Lemma B.1 *Individual consumption rises if and only if current consumption relative to consumption in the other state is sufficiently high.*

For the employed worker, consumption rises if and only if $c(a_w,w)$ relative to $c(a_w,b)$ is sufficiently high,

$$\frac{dc(a_w,w)}{dt} \geq 0 \Leftrightarrow \frac{u'(c(a_w,b))}{u'(c(a_w,w))} \geq 1 - \frac{r-\rho}{s} \Leftrightarrow \frac{c(a_w,w)}{c(a_w,b)} \geq 1/\psi, \tag{52}$$

[20] While this is a standard property of many steady states, the standard solutions (e.g. linearization around the steady state) do not work in our case. This is in part due to the fact that the original stochastic differential equation system (4a) to (4d) - even when stripped of its stochastic part - is not an ordinary differential equation system.

[21] The constant Ψ only serves to make $R_{v,\Psi} \subset \mathbb{R}^3$ a compact set, which we need to obtain global, uniform Lipschitz constants. We shall see below that Ψ has to be chosen larger than $\Psi_0 = \frac{\psi w - b}{(1-\psi)r}$. In this case, however, Ψ does not interfere with the construction.

where

$$\psi \equiv \left(1 - \frac{r-\rho}{s}\right)^{-1/\sigma}. \tag{53}$$

For the unemployed worker, consumption rises if and only if $c(a_b, b)$ relative to $c(a_b, w)$ is sufficiently high,

$$\frac{dc(a_b, b)}{dt} \geq 0 \Leftrightarrow \frac{u'(c(a_b, w))}{u'(c(a_b, b))} \geq 1 - \frac{r-\rho}{\mu} \Leftrightarrow \frac{c(a_b, b)}{c(a_b, w)} \geq \left(1 - \frac{r-\rho}{\mu}\right)^{1/\sigma}. \tag{54}$$

Proof. Rearranging (4a) and (4c) for $dq_s = dq_\mu = 0$ and taking (1) into account gives the results (see app. B.2). Note that in what follows ψ will be used only for r sufficiently small making sure that ψ is a real number. ■

We rely on the following lemma for our proposition. It reads

Lemma B.2 *Relative consumption $c(a, w)/c(a, b)$ is continuously differentiable in wealth a.*

Proof. Consumption levels $c(a, w)$ and $c(a, b)$ are understood as solutions to our ODE system (5). As the latter is well-behaved within the set Q_v from (50), consumption levels are differentiable and thereby continuous in Q_v. This implies the differentiability of $c(a, w)/c(a, b)$.
■

For technical reasons, we also need to make

Assumption 1 *The number of sign changes of the derivative of relative consumption with respect to wealth, i.e. $d(c(a, w)/c(a, b))/da$, in any interval of finite length is finite.*

This assumption is required to rule out "pathological cases". One can construct continuously differentiable functions that change sign infinitely often in a finite neighborhood (think of $x^3 \sin(1/x)$ in a neighborhood of zero). Any economic intuition suggests that such pathological cases are not relevant for our model. We employ this assumption only in this sect. B (and implicitly in sect. C where we refer to sect. B).

Proposition B.3 *Consider a low interest rate, i.e. $0 < r \leq \rho$. Define a threshold level a_w^* by*

$$\frac{u'(c(a_w^*, b))}{u'(c(a_w^*, w))} \equiv 1 - \frac{r-\rho}{s}. \tag{55}$$

For our instantaneous utility function (1), this definition reads

$$c(a_w^*, b) = \psi c(a_w^*, w) \tag{56}$$

where ψ is from (53).

(i) Consumption of employed workers increases if the worker owns a sufficiently low wealth level, $a < a_w^*$. Employed workers with $a > a_w^*$ choose falling consumption paths.

(ii) Consumption of unemployed workers always decreases.

(iii) Consumption of employed workers exceeds consumption of unemployed workers at the threshold a_w^*, i.e. $\psi \leq 1$ in (56) for $r \leq \rho$.

Proof. see app. B.4 ■

B.3 Natural borrowing limit

The subsequent analysis will be facilitated by noting that there is an endogenous "natural" borrowing limit. The idea is similar to Aiyagari's (1994) borrowing limit resulting from non-negative consumption. This limit is derived in the following

Proposition B.4 *Any individual with initial wealth $a \geq -b/r$ will never be able to or willing to borrow more than $-b/r$. Consumption of an unemployed worker at $a = -b/r$ is zero, $c(-b/r, b) = 0$.*

Proof. "willing to": An employed individual with $a \geq -b/r$ will increase wealth for any wealth levels below a_w^* from (55). If a_w^* is larger than $-b/r$ – which we can safely assume – employed workers with wealth below a_w^* increase wealth and are not willing to borrow more than $-b/r$.

"able to": Imagine an unemployed worker had wealth lower than $-b/r$. Even if consumption is equal to zero, wealth would further fall, given that $\dot{a} = ra + b < 0 \Leftrightarrow a < -b/r$. If an individual could commit to zero consumption when employed and if the separation rate was zero, the maximum debt an individual could pay back is $-w/r$. Imagine an unemployed worker succeeded in convincing someone to lend her "money" even though current wealth is below $-b/r$. Then, with a strictly positive probability, wealth will fall below $-w/r$ within a finite period of time. Hence, anyone lending to an unemployed worker with wealth below $-b/r$ knows that not all of this loan will be paid back with positive probability. This cannot be the case in our setup with one riskless asset. Hence, the maximum debt level is b/r and consumption is zero at $a = -b/r$ for an unemployed worker. ■

B.4 Proof of prop. B.3 concerning Keynes-Ramsey rule

B.4.1 Proof of part (i)

- A local result

We first show that consumption $c(a_w, w)$ rises in time for wealth smaller than but close to a_w^*.

Consider relative consumption $\chi(a) \equiv x(a)/y(a)$. By ass. 1, the number of sign changes of $\chi'(a)$ in any interval for a of finite length is finite. We can therefore for any a_0 find an $\varepsilon > 0$ such that $\chi(a)$ is monotonic in $[a_0 - \varepsilon, a_0]$. Exploiting this for a_w^*, whatever the properties of relative consumption, we can always find an ε such that one of the following three cases must hold for $\Omega_\varepsilon \equiv [a_w^* - \varepsilon, a_w^*[$

$$\left.\begin{array}{l}\text{(i)}\\ \text{(ii)}\\ \text{(iii)}\end{array}\right\} \chi'(a)|_{a\in\Omega_\varepsilon} \left\{\begin{array}{c}<\\>\\=\end{array}\right\} 0.$$

Note that we do not make any statement about the derivative in a_w^*. In fact, in case (i) $\chi'(a)|_{a\in a_w^*}$ can be negative or zero, in case (ii), it can be positive or zero.

Lemma B.5 *(a) Consumption of employed workers rises over time for a wealth level $a \in \Omega_\varepsilon$ if and only if case (i) holds,*

$$\frac{dc(a_w(\tau), w)}{d\tau} > 0 \text{ for } a_w(\tau) \in \Omega_\varepsilon \Leftrightarrow \text{case (i) holds.}$$

(b) Consumption $c(a_w(\tau), w)$ falls over time for $a_w(\tau) \in \Omega_\varepsilon$ if and only if (ii) holds.

Proof. (a) By (52), $\frac{dc(a_w(\tau),w)}{d\tau} > 0 \Leftrightarrow c(a_w(\tau),w)/c(a_w(\tau),b) > 1/\psi$. As $c(a_w^*,w)/c(a_w^*,b) = 1/\psi$ at a_w^*, as w and b are parameters and using ass. 1, this is a condition on the derivative of relative consumption with respect to wealth a in Ω_ε: $dc(a_w(\tau),w)/d\tau$ is positive for $a_w(\tau) \in \Omega_\varepsilon$ if and only if case (i) holds.

(b) By (52), consumption falls over time if relative consumption lies below $1/\psi$. This can be the case in Ω_ε only if case (ii) holds. ■

Lemma B.6 *Relative consumption falls in wealth for $a \in \Omega_\varepsilon$, $\chi'(a)|_{a\in\Omega_\varepsilon} < 0$, i.e. case (i) holds.*

Proof. a) Assume that case (ii) holds, i.e. $\chi'(a)|_{a\in\Omega_\varepsilon} > 0$. Then, by lem. B.5, $\frac{dc(a_w(\tau),w)}{d\tau} < 0$ for $a_w(\tau) < a_w^*$. Consumption of unemployed workers would still decrease in time for all wealth levels. In our set Q_v from (50), $\frac{da_w(\tau)}{d\tau} > 0$ and therefore $\frac{dx(a)}{da} < 0$. As $\frac{dc(a_b(\tau),b)}{d\tau} < 0$ and $\frac{da_b(\tau)}{d\tau} < 0$ in Q_v, we know that $\frac{dy(a)}{da} > 0$. As a consequence, $\chi'(a) < 0$. This contradicts the assumption that case (ii) holds and case (ii) can be excluded.

b) Now assume that case (iii) holds, i.e. relative consumption is flat, $\chi'(a)|_{a\in\Omega_\varepsilon\cup a_w^*} = 0$. As $c(a_w^*,w)/c(a_w^*,b) = 1/\psi$, $dc(a_w(\tau),w)/d\tau = 0$ for $a_w(\tau) \in \Omega_\varepsilon$. As $dc(a_b(\tau),b)/d\tau < 0$, relative consumption is not constant – which contradicts the assumption that relative consumption is flat in wealth. As case (iii) is thereby excluded as well, the proof is complete. ■

- A global result

We now complete the proof by a global result on consumption growth.

Lemma B.7 *Consumption $c(a_w,w)$ (a) rises in time for all $a < a_w^*$ and (b) decreases in time for all $a > a_w^*$.*

Proof. (a) Imagine to the contrary of "$c(a_w,w)$ rises in time for all $a < a_w^*$" that there is an interval $]\Gamma_1,\Gamma_2[$ with $\Gamma_2 < a_w^*$ such that this is is the last interval before a_w^* where $c(a_w,w)$ falls in time,

$$dc(a_w(\tau),w)/d\tau < 0, \quad \forall\, \Gamma_1 < a_w(\tau) < \Gamma_2 < a_w^*. \tag{57}$$

We now proceed as in the proof of lem. B.6. As $\frac{da_w(\tau)}{d\tau} > 0$ in Q_v, this would imply that $\frac{dx(a)}{da} < 0$ for $\Gamma_1 < a < \Gamma_2$. We know that $\frac{dy(a)}{da} > 0$ in Q_v. Hence, we would conclude that

$$\chi'(a) < 0, \quad \forall\, \Gamma_1 < a < \Gamma_2. \tag{58}$$

By (52), the assumption in (57) would hold if and only if relative consumption $\frac{c(a_w,w)}{c(a_w,b)}$ is below $1/\psi$ for $\Gamma_1 < a < \Gamma_2$: $\frac{dc(a_w(\tau),w)}{d\tau} < 0 \Leftrightarrow \frac{c(a_w(\tau),w)}{c(a_w(\tau),b)} < 1/\psi$. As $\frac{x(a)}{y(a)}$ is continuous in wealth by lem. B.2 and as case (i) holds by lem. B.6, $\frac{x(a)}{y(a)}$ can be smaller than $1/\psi$ only if there is some range $]\Gamma_3,\Gamma_2[$ in which $\chi'(a) > 0$. (An example of such a path is shown in fig. 3.) This is a contradiction to the conclusion in (58). Hence, consumption must rise in time for all $a < a_w^*$.

(b) This proof is in analogy to the proof of (a). ■

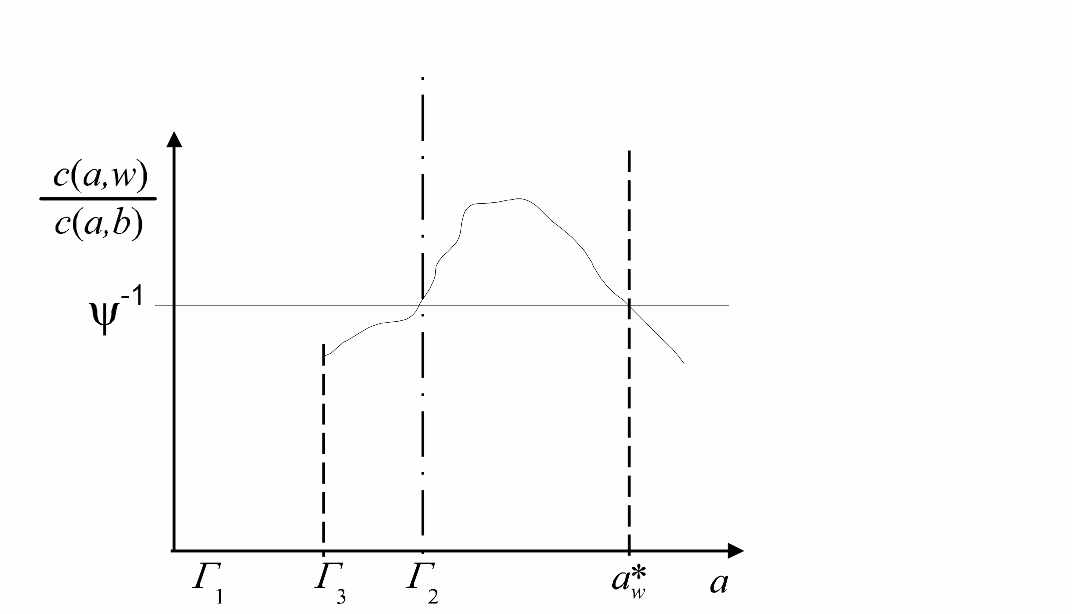

Figure 3 *An example for relative consumption* $\chi(a) \equiv \frac{x(a)}{y(a)}$

B.4.2 Intermediary steps

Before we prove the rest of prop. B.3, we need some further intermediary results – which, however, are of some interest in their own right. Given that marginal utility from (1) is positive and decreasing, $u'(c) > 0$ and $u''(c) < 0$, we can establish that $x(a) > y(a)$, i.e. consumption in the state of employment is larger than in the state of unemployment, keeping wealth constant. We prove in passing that the value functions $V(a,z)$ are strictly concave in wealth a.

Lemma B.8 *Consumption rises in wealth,* $c_a(a,z) > 0$.

Proof. Prop. B.3 (i) shows that $dc(a_w(\tau), w)/d\tau > 0$ in Q_v. As $da_w(\tau)/d\tau > 0$ as well, the derivative $dx(a)/da$ in (5) is positive in Q_v. ■

Lemma B.9 *As marginal utility from consumption is positive, the value function* $V(a,z)$ *rises in wealth,* $V_a(a,z) > 0$.

Proof. The first-order condition for optimal consumption is given by (45) in the Referees' appendix and reads

$$u'(c(a,z)) = V_a(a,z). \tag{59}$$

As marginal utility is positive by (1), the value function rises in wealth. ■

Lemma B.10 *As* $u''(c) < 0$ *and as consumption rises in* a *by lemma B.8, the value function is strictly concave in* a.

Proof. The partial derivative of the first-order condition with respect to wealth implies

$$u''(c(a,z))\, c_a(a,z) = V_{aa}(a,z). \tag{60}$$

As $u''(c(a,z)) < 0$ from the concavity of (1) and $c_a(a,z)$ is positive by lem. B.8, $V_{aa}(a,z)$ must be negative. With lem. B.9, the value function is strictly concave. ■

Lemma B.11 *The shadow price for wealth is higher in the state of unemployment,* $V_a(a,b) > V_a(a,w)$.

Proof. The derivation of the Keynes-Ramsey rule gives us (see app. B.1)

$$\begin{aligned}&(\rho - r) V_a(a,z) - s(z)[V_a(a,b) - V_a(a,w)] - \mu(z)[V_a(a,w) - V_a(a,b)] \\ &= [ra + z - c(a,z)] V_{aa}(a,z).\end{aligned}$$

In state $z = w$, this means

$$(\rho - r) V_a(a,w) - s(z)[V_a(a,b) - V_a(a,w)] = [ra + w - x(a)] V_{aa}(a,w). \tag{61}$$

Given the region we are interested in (where $ra+w-x(a) > 0$) and given lemma B.10, the right-hand side is negative. Hence, the left-hand side must be negative as well. As $(\rho - r) V_a(a,w)$ is positive due to $r < \rho$, the second term must be negative. This is the case only for $V_a(a,b) > V_a(a,w)$. ■

Lemma B.12 *Consumption of the employed worker is higher than consumption of the unemployed worker,* $x(a) > y(a)$.

Proof. As $V_a(a,b) > V_a(a,w)$, the first-order condition implies $u'(y(a)) > u'(x(a))$. As the marginal utility is decreasing, $x(a) > y(a)$. ■

B.4.3 Proof of parts (ii) and (iii)

(ii) By (54), $dc(a_b(\tau), b)/d\tau < 0 \Leftrightarrow u'(c(a_b(\tau), w)) < \varkappa u'(c(a_b(\tau), b))$ where $\varkappa \equiv 1 - \frac{r-\rho}{\mu} \geq 1$ as $r \leq \rho$. As $u'(c(a_b(\tau), w)) < u'(c(a_b(\tau), b))$ with $c(a_b(\tau), w) > c(a_b(\tau), b)$ from lem. B.12, this condition always holds.

(iii) This follows from solving (55) for relative consumption.

C Existence of an optimal consumption path

This appendix provides a proof for the existence of a path $c(a,z)$ as depicted in fig. 1. We now introduce an auxiliary TSS (aTSS) in order to capture v. In analogy to the TSS Θ from (7), this point is defined by

$$\Theta_v \equiv (a_w^*, c_v(a_w^*, w)),$$

i.e. the wealth level a_w^* is unchanged but the consumption level is "a bit lower" than in the TSS. In the TSS, the consumption level is on the zero-motion line, i.e. $c(a_w^*, w) = ra_w^* + w$. In the aTSS, the consumption level is on the line $ra + w - v$ and therefore given by $c_v(a_w^*, w) = ra_w^* + w - v$. Let us now consider the following

Definition C.1 *(Optimal consumption path) A* consumption path *is a solution* $(a, c(a,w), c(a,b))$ *of the ODE-system (5) for the range* $-b/r \leq a \leq a_w^*$ *in* $R_{v,\Psi}$ *from (51) with terminal condition* $(a_w^*, c_v(a_w^*, w), c_v(a_w^*, b))$. *In analogy to the aTSS and to (56), the terminal condition satisfies* $c_v(a_w^*, w) = ra_w^* + w - v$ *and* $c_v(a_w^*, b) = \psi c_v(a_w^*, w)$ *for an arbitrary* $a_w^* > -b/r$. *An* optimal consumption path *is a consumption path which in addition satisfies* $c(-b/r, b) = 0$.

App. C.1 then proves

Theorem C.2 *There is an optimal consumption path.*

This establishes that we can continue in our analysis by taking the existence of a path $c(a,z)$ as given. Intuitively speaking, i.e. looking at v as very small constant close to zero, we know that there are paths $c(a,w)$ and $c(a,b)$ as drawn in fig. 1. The approximation implied by the auxiliary TSS is very small compared to any measurement error in the data. Values of $v = 10^{-3}$ worked perfectly in numerical solutions.

For simple reference in what follows and to simplify notation, define

$$x(a) \equiv c(a,w), \quad y(a) \equiv c(a,b), \tag{62}$$

and express the reduced form (5) as

$$\dot{x}(a) = \frac{r - \rho + s\left[\left(\frac{x(a)}{y(a)}\right)^{\sigma} - 1\right]}{ra + w - x(a)} \frac{x(a)}{\sigma}, \tag{63a}$$

$$\dot{y}(a) = \frac{r - \rho - \mu\left[1 - \left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra + b - y(a)} \frac{y(a)}{\sigma}. \tag{63b}$$

C.1 Proof of theo. C.2 - existence of an optimal consumption path

C.1.1 Preliminaries

The natural borrowing limit implies that any solution to (63) must satisfy

$$y(-b/r) = 0. \tag{64}$$

In what follows, we will use classical theorems for initial value problems for ODEs. Currently, we have formulated our system (63) as a terminal value problem, since the definition of the optimal consumption path in def. C.1 uses a terminal condition at the end of the interval $[-b/r, a_w^*]$ under consideration. Using the notation from (62), this terminal condition can be written in compact form as

$$\Phi \equiv \Phi_v(\hat{a}) = (\hat{a}, x_v(\hat{a}), y_v(\hat{a})). \tag{65}$$

Note that Φ depends on v

For ease of notation and to help intuition, we shall now recast the problem into a classical initial value problem, i.e. we will require the value Φ to be attained at the fixed beginning $\tau = 0$ of an interval $[0, \tau^*]$, on which we study the problem. To this end, it is more useful to work with an autonomous system. Hence, we rewrite (63) by including $m(a) = a$ as third variable which "replaces" wealth a, which now purely serves as a parameter, i.e. as the independent variable. By using (62), this gives the system

$$\dot{m}(a) = 1,$$

$$\dot{x}(a) = \frac{r - \rho + s\left[\left(\frac{x(a)}{y(a)}\right)^{\sigma} - 1\right]}{rm(a) + w - x(a)} \frac{x(a)}{\sigma},$$

$$\dot{y}(a) = \frac{r - \rho - \mu\left[1 - \left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{rm(a) + b - y(a)} \frac{y(a)}{\sigma}.$$

Now define $\tau \equiv \hat{a} - a$, $x_1(\tau) \equiv m(\hat{a} - \tau)$, $x_2(\tau) \equiv x(\hat{a} - \tau)$, $x_3(\tau) \equiv y(\hat{a} - \tau)$. Then, $\frac{d}{d\tau} x_1(\tau) \equiv \dot{x}_1(\tau) = \frac{d}{d\tau} m(\hat{a} - \tau) = \frac{d}{d[\hat{a} - a]} m(a) = -\frac{d}{da} m(a) = -\dot{m}(a)$. Doing the same for x

and y, the "inverted" autonomous system therefore reads

$$\dot{x}_1(\tau) = -1, \tag{66a}$$

$$\dot{x}_2(\tau) = -\frac{r-\rho+s\left[\left(\frac{x_2(\tau)}{x_3(\tau)}\right)^{\sigma}-1\right]}{rx_1(\tau)+w-x_2(\tau)}\frac{x_2(\tau)}{\sigma}, \tag{66b}$$

$$\dot{x}_3(\tau) = -\frac{r-\rho-\mu\left[1-\left(\frac{x_3(\tau)}{x_2(\tau)}\right)^{\sigma}\right]}{rx_1(\tau)+b-x_3(\tau)}\frac{x_3(a)}{\sigma}, \tag{66c}$$

where now $\dot{x}_i$ denotes the derivative of $x_i(\tau)$ with respect to τ, $i=1,2,3$.

Definition C.3 *Given (66) and for $\tau \geq 0$, let $X(\tau;\Phi) = (x_1(\tau), x_2(\tau), x_3(\tau))$ denote the solution of (66) started at $X(0;\Phi) = \Phi \in R_{v,\Psi}$ from (65) where $-b/r \leq \hat{a} \leq \frac{\Psi+v-w}{r}$. For later use, we also introduce the notation $x_i(\tau) = x_i(\tau;\Phi)$, $i=1,2,3$.*

By passing from (63) to (66) we have reverted the time-direction – more precisely, in our setting, the wealth-direction – and turned a non-autonomous system into an autonomous one by including the independent variable as an additional component of the solution. Thus, the curve $a \mapsto (a, x(a), y(a))$ with terminal value $x(\hat{a}) = x_v(\hat{a})$, $y(\hat{a}) = y_v(\hat{a})$ is equal to the curve $\tau \mapsto X(\tau;\Phi)$ with $\Phi = \Phi(\hat{a})$, which is the solution of an initial value problem in the classical sense. However, the parametrization is reverted in the sense that in the former case we start at the left endpoint ("left" in the sense of the smallest value of the a-component) and end in the right endpoint, whereas in the latter case we start at the right endpoint and end in the left one. In particular, the absolute value of the speed along the curve is equal, but the direction is reversed.

C.1.2 Continuity of the solution in initial values

In order to be able to apply classical theorems, we need finite derivatives on the right-hand side of an ODE system. The right-hand side of the ODE (63), however, exhibits singularities at the boundary $y = ra+b$ of Q_v. This is of particular importance as the definition of the optimal consumption path in Definition C.1 uses $y(-b/r) = 0$ – which lies on this boundary. We obtain finite derivatives by (i) a coordinate transformation and by (ii) (temporarily) reducing the set on which we are interested in a solution by demanding that $y \geq \varepsilon$. We will later show how this reduction can then be removed again by passing $\varepsilon \to 0$.

Lemma C.4 *(Coordinate transformation) Let $x(a)$ and $y(a)$ be solutions of (63). The mapping $a \mapsto y(a)$ is bijective. Change variables $a = a(y)$ and consider x and a as functions of y. Then*

$$x'(y) \equiv \frac{dx(y)}{dy} = \frac{r-\rho+s\left[\left(\frac{x(y)}{y}\right)^{\sigma}-1\right]}{r-\rho-\mu\left[1-\left(\frac{y}{x(y)}\right)^{\sigma}\right]}\frac{x(y)}{y}\frac{ra(y)+b-y}{ra(y)+w-x(y)}, \tag{67a}$$

$$a'(y) \equiv \frac{da(y)}{dy} = \frac{ra(y)+b-y}{r-\rho-\mu\left[1-\left(\frac{y}{x(y)}\right)^{\sigma}\right]}\frac{\sigma}{y}. \tag{67b}$$

Proof. Since $\dot{y}(a) > 0$, y is a bijective function of a. As $a'(y) = \frac{1}{\dot{y}(a)}$, we obtain the second equation by inserting (63b). The first equation follows from "dividing (63a) by (63b)". ■

We are going to avoid the singularity at $y(-b/r) = 0$ by temporarily requiring these properties only to hold "up to an arbitrarily small number ε". We do this by considering the domain $R_{\varepsilon,v,\Psi}$ as given in the following

Definition C.5 *Fix a numbers $\varepsilon > 0$ and define*

$$R_{\varepsilon,v,\Psi} = R_{v,\Psi} \cap \left\{(a,x,y) \in \mathbb{R}^3 \,|\, y \geq \varepsilon\right\}. \tag{68}$$

This definition implies that we temporarily replace the requirement that $y(-b/r) = 0$ by $y(a) = \varepsilon$ for some $-b/r \leq a \leq -b/r + \varepsilon/r$.

Lemma C.6 *The right-hand side given in (67) is uniformly Lipschitz on $R_{\varepsilon,v,\Psi}$.*

Proof. Consider the right-hand side of (67a). The only possible points, where the Lipschitz constant can explode, are when the denominators in the right-hand side become 0 or when a term under a fractional power (i.e. with exponent σ) becomes 0. In $R = R_{\varepsilon,v,\Psi}$, y is uniformly bounded away from 0 and x is uniformly bounded away from $ra + w$. Moreover, note that $r - \rho - \mu\left[1 - \left(\frac{y}{x}\right)^{\sigma}\right] = 0$ if and only if $\left(\frac{y}{x}\right)^{\sigma} = 1 - \frac{r-\rho}{\mu}$. Now $1 - \frac{r-\rho}{\mu} > 1$ by the assumption that $r < \rho$. On the other hand, $y < x$, implying that $\left(\frac{y}{x}\right)^{\sigma} < 1$. Consequently, all the denominators are uniformly bounded away from 0.

For the fractional powers, note that $x/y > 1$ is trivially uniformly bounded away from 0. As $x \leq \Psi$,

$$\frac{y}{x} > \frac{\epsilon}{\Psi}$$

is uniformly bounded away from 0 on $R_{\varepsilon,v,\Psi}$. This shows that (67a) is uniformly Lipschitz.

The same arguments show that the right-hand side of (67b) is uniformly Lipschitz, too. ■

Since the right hand side of (67) is uniformly Lipschitz, we can now apply the classical theory of ODEs. For instance, we have existence and uniqueness of the solution by the Picard-Lindelöf theorem, see Mattheij and Molenaar (2002, th. II.2.3, th. II.3.1). Moreover, the solution will be continuous as a function of the initial value, see, again, Mattheij and Molenaar (2002, th. II.4.7). In the lemma below, we will see how this even implies the corresponding properties for the non-transformed system (66).

Lemma C.7 *(Continuity in initial values) Consider the set $R = R_{\varepsilon,v,\Psi}$ from (68) and the solution $X(\tau;\Phi)$ from Definition C.3 with initial condition Φ given in (65). The solution $X(\tau;\Phi)$ depends continuously on its initial values Φ. More precisely, there is a constant $L > 0$ and an increasing map $\kappa : [0,\infty[\to [0,\infty[$ (a* modulus of continuity*) with $\lim_{t\searrow 0}\kappa(t) = \kappa(0) = 0$ such that*

$$\|X(\tau_1;\Phi_1) - X(\tau_2;\Phi_2)\| \leq L\|\Phi_1 - \Phi_2\| + \kappa(|\tau_1 - \tau_2|),$$

provided that $\Phi_1, \Phi_2 \in R$ and $X(\tau;\Phi_i) \in R$ for all $0 \leq \tau \leq \max(\tau_1,\tau_2)$, $i = 1, 2$. Here, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^3$.

Proof. By classical results from the theory of ordinary differential equations, see for instance Mattheij and Molenaar (2002, th. II.4.7), the solution of an ODE-system depends continuously on the initial data as long as the right-hand side is uniformly Lipschitz. More precisely, let $Y(\tau;\Phi)$ denote the solution of an ODE with uniformly Lipschitz right-hand side (with Lipschitz constant C), started at $Y(\tau_0;\Phi) = \Phi$, then

$$\|Y(\tau;\Phi_1) - Y(\tau;\Phi_2)\| \leq \exp\left(C(\tau - \tau_0)\right)\|\Phi_1 - \Phi_2\|.$$

Now consider the transformed system $(a(y), x(y))$ from (67). By Lemma C.6, the right-hand side is uniformly Lipschitz. The solution of (67) therefore depends continuously on its initial data (a_0, x_0). It is then obvious that the trajectory $(a(y), x(y), y)$ depends continuously on (a_0, x_0, y_0). As system (67) is a reparameterized version of (63), the solution $(a, x(a), y(a))$ to (63) from def. C.1 is also continuous in its boundary conditions – even though the right hand side of (63) is not uniformly Lipschitz. Similarly, as (66) is just a reparameterization of (63), the solution $X(\tau;\Phi)$ to (66) from def. C.3 is also continuous in its initial condition Φ.

In order to get the estimate, we now consider the ODE (66) and note that we only consider it on the compact set $R_{\varepsilon,v,\Psi}$. In the parametrization by y given in (67), y is the independent variable, i.e. plays the role of τ in the above estimate. By compactness of $R_{\varepsilon,v,\Psi}$, y only runs through a bounded set, therefore we can rewrite the constant in the above inequality as $\exp(C(y-y_0)) \leq L$ for some suitable $L>0$.

Given $\Phi \in R_{\varepsilon,v,\Psi}$. Then $a_w^* \leq \frac{\Psi-w+v}{r}$, which implies that the solution $X(\tau;w)$ can only stay inside $R_{\varepsilon,v,\Psi}$ until time $\tau = \frac{\Psi-w+v+b}{r}$, at most. Consider

$$D = \{(\tau,\Phi) \in [0,\infty[\times R_{\varepsilon,v,\Psi} \,|\, X(\tau;\Phi) \in R_{\varepsilon,v,\Psi}\}.$$

Then D is a closed subset of $\left[0, \frac{\Psi-w+v+b}{r}\right] \times R_{\varepsilon,v,\Psi}$, implying that D is compact. Consequently, $X : D \to R_{\varepsilon,v,\Psi}$ is uniformly continuous, which implies the existence of a modulus of continuity κ with

$$\|X(\tau_1;\Phi_1) - X(\tau_2;\Phi_2)\| \leq \kappa(|\tau_1-\tau_2| + \|\Phi_1-\Phi_2\|).$$

The inequality in the lemma then follows by the triangle inequality. ■

C.1.3 Continuity of the first hitting-wealth in initial values

While we have shown in the previous section that the solutions to all systems (63), (66) and (67) are continuous in initial values, this does not automatically imply that the solutions will be continuous on the boundary of the domain we are interested in, in the sense that the place where the solution leaves the domain R might not depend continuously on the initial data. This will now be proved in this section.

In the proofs and also in a later step, we will use the following

Definition C.8 *(First hitting-wealth) Consider the set $R_{\varepsilon,v,\Psi}$ from (68) and the solution $X(\tau;\Phi)$ to the system (66). Consider the path $y(a)$ that corresponds to $x_2(\tau)$ of this solution. Then we define $\hat{a}_{1st} = f(\hat{a})$ as the "first hitting-wealth" (in analogy to first hitting-time), i.e. the wealth level where the path $y(a)$ hits any boundary of $R_{\varepsilon,v,\Psi}$ for the first time. Similarly denote $\tau(\Phi) \equiv \inf\{\tau \geq 0 \,|\, X(\tau;\Phi) \in \partial R_{\varepsilon,v,\Psi}\}$ and $F(\Phi) \equiv X(\tau(\Phi);\Phi)$.*

We know that $\hat{a}_{1st}$ exists because in the set $R_{\varepsilon,v,\Psi}$ the derivatives in (66) are well-defined and a solution therefore exists. Notice that $\hat{a}_{1st}$ equals the first component of $F(\Phi(\hat{a}))$.

We also need

Definition C.9 *Let $N \subset R_{\varepsilon,v,\Psi}$ with*

$$N = \left\{ \Phi(\hat{a}) \,\middle|\, \hat{a} \in \left[-\frac{b}{r}, \frac{\psi[w-v]-b}{r[1-\psi]}\right] \right\}$$

be the set of all potential initial conditions from (65) for a solution in the sense of def. C.1. Here we implicitly assume that Ψ is large enough that indeed $N \subset R_{\varepsilon,v,\Psi}$.[22] Define M as

$$M = M_1 \cup M_2 \cup M_3 \subset R_{\varepsilon,v,\Psi} \tag{69}$$

with

$$M_1 = \{(a,x,y) \in R_{\varepsilon,v,\Psi} \,|\, y = ra+b\},$$
$$M_2 = \{(a,x,y) \in R_{\varepsilon,v,\Psi} \,|\, a = -b/r\},$$
$$M_3 = \{(a,x,y) \in R_{\varepsilon,v,\Psi} \,|\, y = \varepsilon\}.$$

This set will turn out to be the set of all potential first hitting-wealths.

[22] This is the only necessary condition on Ψ for the construction to work. In the sequel, we shall assume this condition without further notice.

Since we know that $x > y$, the trajectory will not hit the boundary of R at the part $\{x = y\}$. Therefore, we have the

Corollary C.10 *$F : N \to M$ is a well-defined map, i.e. for every $\Phi \in N$, the corresponding solution path $X(\tau; \Phi)$ exists and stays in $R_{\varepsilon,v,\Psi}$ until it finally hits M (and no other boundary of $R_{\varepsilon,v,\Psi}$).*

Before formulating the main lemma of this section, let us first derive a simple bound on the derivative $\dot{y}(a)$ of the consumption of the unemployed.

Lemma C.11 *For (a, x, y) in the interior of Q_v from (50), we have*

$$\dot{y}(a) \geq \frac{r - \rho}{ra + b - y(a)} \frac{y(a)}{\sigma}.$$

Proof. By (63b) we have

$$\begin{aligned}
\dot{y}(a) &= \frac{r - \rho - \mu\left[1 - \left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra + b - y(a)} \frac{y(a)}{\sigma} \\
&= \left(\frac{r - \rho}{ra + b - y(a)} - \frac{\mu\left[1 - \left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra + b - y(a)} \right) \frac{y(a)}{\sigma} > \frac{r - \rho}{ra + b - y(a)} \frac{y(a)}{\sigma}.
\end{aligned}$$

The last inequality follows from the fact that $\frac{\mu\left[1 - \left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra+b-y(a)}$ is negative (and therefore $-\frac{\mu\left[1 - \left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra+b-y(a)}$ is positive) as $ra + b - y(a)$ is negative in the interior of Q_v. ■

The key result in this section is presented in

Lemma C.12 *The map $F : N \to M$ is continuous.*

Proof. We need to prove that for every $\Phi \in N$ and every $\delta > 0$ there is an $\eta > 0$ such that

$$\|\Phi_0 - \Phi\| < \eta \Longrightarrow \|F(\Phi_0) - F(\Phi)\| < \delta. \tag{70}$$

We start the proof by fixing Φ_0, $\Phi \in N$ such that $\|\Phi_0 - \Phi\| < \eta$ for some $\eta > 0$. Let us first *assume* that $\tau(\Phi_0) \leq \tau(\Phi)$. By the triangle inequality and Lemma C.7, we have

$$\begin{aligned}
\|X(\tau(\Phi_0); \Phi_0) - X(\tau(\Phi); \Phi)\| &\leq \|X(\tau(\Phi_0); \Phi_0) - X(\tau(\Phi_0); \Phi)\| + \\
&\quad + \|X(\tau(\Phi_0); \Phi) - X(\tau(\Phi); \Phi)\| \\
&\leq L_1 \|\Phi_0 - \Phi\| + \kappa(|\tau(\Phi_0) - \tau(\Phi)|),
\end{aligned} \tag{71}$$

for a constant $L_1 > 0$ and the modulus of continuity κ. In order to get an estimate for $|\tau(\Phi_0) - \tau(\Phi)|$, we have to distinguish between three different cases.

Case (i): $F(\Phi_0) \in M_1$.

By Lemma C.11, there are constants $L_2, \ell_2 > 0$ such that $\dot{y} \geq L_2$ for $|y - (ra + b)| \leq \ell_2$. More precisely, we can choose $\ell_2 > 0$ freely and obtain the bound for $L_2 = \frac{1}{\ell_2} \frac{(\rho - r)\varepsilon}{\sigma}$. If $L_1 \eta \leq \ell_2$, we can bound the absolute value of the derivative of $x_3(\tau; \Phi)$ from below by L_2 (for $t \geq \tau(\Phi_0)$). This implies that the path $X(\tau; \Phi)$ hits M_1 before time $\tau(\Phi_0) + \tau$ for

$$\tau(L_2 - r) = \ell_2 \iff \tau = \frac{\ell_2}{L_2 - r},$$

unless it hits another boundary of $R_{\varepsilon,v,\Psi}$ before that. Inserting into (71), this gives the estimate

$$\|F(\Phi_0) - F(\Phi)\| \leq L_1\eta + \kappa\left(\frac{\ell_2}{L_2 - r}\right).$$

Choosing $\ell_2 = L_1\eta$, the bound is smaller than δ provided that

$$\kappa\left(\frac{L_1}{\frac{C}{L_1\eta} - r}\eta\right) + L_1\eta < \delta, \tag{72}$$

where $C \equiv \frac{(\rho-r)\varepsilon}{\sigma}$. Note that the left hand side in (72) converges to zero for $\eta \to 0$, therefore we can find an $\eta_0(\delta) > 0$ (only depending on the constants C, L_1 and r and the modulus of continuity κ, but not on Φ_0 or Φ) such that the desired inequality (70) holds for $\eta < \eta_0$. We have tacitly assumed that $L_2 = C/\ell_2 = \frac{C}{L_1\eta} > r$, which can be realized by choosing η small enough.

Case (ii): $F(\Phi_0) \in M_2$.
Let $\hat{a}$ denote the first component of Φ, and $\hat{a}_0$ the first component of Φ_0. Note that $x_1(\tau;\Phi) = \hat{a} - \tau$, for every $\tau \geq 0$. Since $X(\tau(\Phi_0);\Phi_0) \in M_2$, we have $-b/r = x_1(\tau(\Phi_0);\Phi_0) = \hat{a}_0 - \tau(\Phi_0)$, implying that $\tau(\Phi_0) = \hat{a}_0 + b/r$. On the other hand, $x_1(\tau(\Phi);\Phi) \geq -b/r$, implying that $\tau(\Phi) \leq \hat{a} + b/r$. Combining these two results, we obtain

$$|\tau(\Phi_0) - \tau(\Phi)| = \tau(\Phi) - \tau(\Phi_0) \leq \hat{a} - \hat{a}_0 \leq \|\Phi_0 - \Phi\|.$$

Consequently, the inequality (71) implies

$$\|F(\Phi_0) - F(\Phi)\| \leq L_1\|\Phi_0 - \Phi\| + \kappa(\|\Phi_0 - \Phi\|) \leq L_1\eta + \kappa(\eta),$$

and (70) holds for η small enough such that

$$L_1\eta + \kappa(\eta) < \delta. \tag{73}$$

Case (iii): $F(\Phi_0) \in M_3$.
Since $x_3(\tau(\Phi_0);\Phi_0) = \varepsilon$, we have $0 \leq x_3(\tau(\Phi_0);\Phi) - \varepsilon \leq L_1\eta$. By Lemma C.11, we can find a constant $L_3 > 0$ such that $\dot{y} \geq L_3$ on $R_{\varepsilon,v,\Psi}$ – note that L_3 depends on ε. Thus, $X(s;\Phi)$ will hit the boundary M_3 before time $\tau(\Phi_0) + \tau$ with $\tau = L_1\eta/L_3$, unless it hits another boundary of $R_{\varepsilon,v,\Psi}$ before. In any case, $|\tau(\Phi_0) - \tau(\Phi)| \leq L_1\eta/L_3$, and we obtain

$$\|F(\Phi_0) - F(\Phi)\| \leq L_1\eta + \kappa\left(\frac{L_1}{L_3}\eta\right),$$

and (70) is satisfied for

$$L_1\eta + \kappa\left(\frac{L_1}{L_3}\eta\right) < \delta. \tag{74}$$

Choosing η small enough that both (72) and (73) and (74) are satisfied, settles the proof for $\tau(\Phi_0) \leq \tau(\Phi)$. Notice that none of the conditions (72), (73) and (74) depends on Φ_0. Therefore, in the other case $\tau(\Phi_0) \geq \tau(\Phi)$, we can just revert the rôles of Φ and Φ_0 and obtain the same results in cases (i), (ii) and (iii). ■

C.1.4 Existence of a solution

This section proves our main result formulated in Theorem C.2.

Proof. Fix some $\varepsilon > 0$ and consider $R_{\varepsilon,v,\Psi}$. By an intermediate value theorem applied to $F : N \to M$, we will obtain a point or points $\Phi \in N$ such that $F(\Phi) \in M_3$ as used in (69), i.e. $x_3(\tau(\Phi);\Phi) = \varepsilon$ provided that we can show the existence of points (that could be called upper and lower bounds) $\Phi_v^{\min}, \Phi_v^{\max} \in N$ with $F(\Phi_v^{\min}) \in M_2$ and $F(\Phi_v^{\max}) \in M_1$. (Note that $F = F_\varepsilon$ and all the $M_i = M_i(\varepsilon)$, $i = 1, 2, 3$, depend on ε and v, but not on Ψ, provided that Ψ is large enough.)

Choose

$$\Phi_v^{\min} = \Phi(-b/r) = (-b/r, w - b - v, \psi\,[w - b - v]), \quad \Phi_v^{\max} = \Phi\left(\frac{\psi(w-v) - b}{(1-\psi)r}\right).$$

By construction, both $\Phi_v^{\min}$ and $\Phi_v^{\max}$ are contained in N. Moreover, we trivially have $F_\varepsilon(\Phi_v^{\min}) \in M_2(\varepsilon)$, $F_\varepsilon(\Phi_v^{\max}) \in M_1(\varepsilon)$ for every $\varepsilon > 0$ small enough. Note, in particular, that Lemma C.12 also implies continuity of F in the boundary points $\Phi_v^{\min}$ and $\Phi_v^{\max}$ of N. Therefore, the image set $F_\varepsilon(N)$ is a connected set, with non-empty intersection with both M_1 and M_2. Since the distance

$$\operatorname{dist}(M_1, M_2) = \inf\left\{\|\Phi_1 - \Phi_2\| \mid \Phi_1 \in M_1,\ \Phi_2 \in M_2\right\} = \frac{\varepsilon}{r} > 0,$$

we may conclude that $F_\varepsilon(N) \cap M_3(\varepsilon) \neq \emptyset$. This establishes that there must be a Φ such that $F_\varepsilon(\Phi) \in M_3$. In words, there is an initial condition $\Phi(\hat{a})$ such that the path $(a, x(a), y(a))$ hits the boundary at $y = \varepsilon$.

Now define

$$N_3(\varepsilon) \equiv F_\varepsilon^{-1}(M_3(\varepsilon)) = \left\{\Phi \in N \mid F_\varepsilon(\Phi) \in M_3(\varepsilon)\right\}.$$

By continuity of $F_\varepsilon : N \to M(\varepsilon)$, the bounded set $N_3(\varepsilon)$ is closed and thus compact. Moreover, the family $(N_3(\varepsilon))_{\varepsilon>0}$ is directed in the sense that

$$0 < \varepsilon_2 < \varepsilon_1 \Longrightarrow N_3(\varepsilon_2) \subset N_3(\varepsilon_1).$$

By standard results from topology, the intersection of a directed family of non-empty, compact sets is non-empty, i.e.

$$N_3(0) \equiv \bigcap_{\varepsilon>0} N_3(\varepsilon) \neq \emptyset.$$

Indeed, take a decreasing sequence $(\varepsilon_n)_{n\geq 1}$ of positive numbers converging to zero. For every n choose some $\Phi_n \in N_3(\varepsilon_n)$. By compactness of the largest set $N_3(\varepsilon_1)$, we can find a subsequence n_k such that $(\Phi_{n_k})_{k\geq 1}$ converges to some Φ. Note that $\Phi \in N_3(\varepsilon_{n_k})$ for every k, since $\Phi = \lim_{l\to\infty,\, l\geq k} \Phi_{n_l}$ and each such Φ_{n_l} lies in the closed set $N_3(\varepsilon_{n_k})$. Now choose any $\varepsilon > 0$ and pick a k such that $\varepsilon_{n_k} < \varepsilon$. Then $\Phi \in N_3(\varepsilon_{n_k}) \subset N_3(\varepsilon)$, implying that $\Phi \in \bigcap_{\varepsilon>0} N_3(\varepsilon)$.

We claim that every element $\Phi \in N_3(0)$ corresponds to an aTSS. Indeed, the path $(a, x(a), y(a))$ with terminal value $(\hat{a}, \hat{x}, \hat{y}) = \Phi$ (corresponding to the path $X(\tau;\Phi)$) satisfies the ODE (63) on $]-b/r, \hat{a}]$. Moreover, it starts at N by construction, and for every $\varepsilon > 0$, it takes on the value ε somewhere on the interval $]-b/r, -b/r + \varepsilon[$. Thus, using monotonicity of y, we may conclude that

$$\lim_{a \searrow -b/r} y(a) = 0.$$

This establishes that there is an initial condition $\Phi(\hat{a})$ such that the path $y(a)$ hits the boundary at $y = 0$ in the sense that $y(-b/r) = 0$. ■

Note that it is essential for the proof of Theorem C.2 that the trajectory $X(\tau;\Phi)$ – or, equivalently, $(a, x(a), y(a))$ – does not depend on ε, which only determines "how long" we observe the trajectory. This means that we observe the trajectory $X(\tau;\Phi)$ for $0 \leq \tau \leq \tau(\Phi)$, with the hitting time $\tau(\Phi)$ obviously depending on ε. Therefore, we can, for fixed $\Phi \in N_3(0)$, easily take the limit $\varepsilon \to 0$, which means that we take the limit in $\tau(\Phi)$, but do not change the trajectory itself. As a consequence, the ODE is automatically satisfied for the limit, at least for $0 \leq \tau < \lim_{\varepsilon\to 0} \tau(\Phi)$.

Let us illustrate why we had to use the specific properties of the dynamic system (66) in the proof of lem. C.12. Continuity in initial conditions does not imply continuity of "first hitting values" in general. Indeed, the first hitting times are inherently non-continuous functionals, even if both the paths and the set, which determines the hitting times, are smooth.

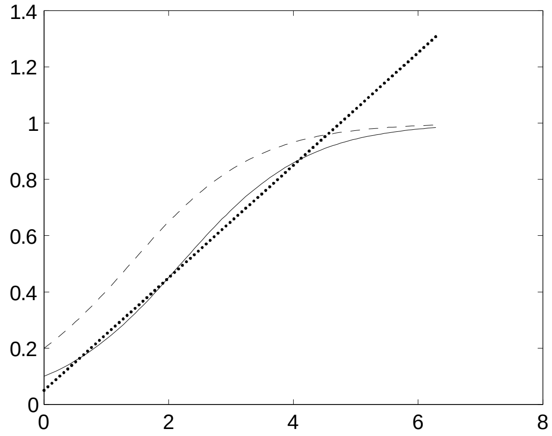

Figure 4 *Non-continuity of the first hitting time*

To see this most clearly, consider the differential equation $\dot{z}(t) = (1 - z(t)) z(t)$ whose solution is $z(t) = \left(1 + \left(z_0^{-1} - 1\right) e^{-t}\right)^{-1}$. This solution is continuous in the initial level z_0 (for $z_0 > 0$ which we assume) and the solution is plotted for $z_0 \in \{0.1, 0.2\}$ in fig. 4. Now consider the first-hitting time on the straight line $0.05 + t/5$ as drawn. Obviously, this time is not continuous in the initial values z_0.

D Properties of the wealth distribution

D.1 A density gives a density

Given the definition of $I(t)$ in prop. 5.1, write the time derivative as

$$\frac{d}{d\tau} I(\tau) = \frac{d}{d\tau} \int_{-\infty}^{\infty} p(a, w, \tau) + p(a, b, \tau)\, da = \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} p(a, w, \tau) + \frac{d}{d\tau} p(a, b, \tau)\right] da.$$

Using the partial differential equations (25), we get

$$\begin{aligned}
&\frac{d}{d\tau} I(\tau) \\
&= \int_{-\infty}^{\infty} \left[-\{ra + w - c(a, w)\} \frac{\partial}{\partial a} p(a, w, \tau) - \left\{r - \frac{\partial}{\partial a} c(a, w) + s\right\} p(a, w, \tau) + \mu p(a, b, \tau)\right] da \\
&+ \int_{-\infty}^{\infty} \left[-\{ra + b - c(a, b)\} \frac{\partial}{\partial a} p(a, b, \tau) - \left\{r - \frac{\partial}{\partial a} c(a, b) + \mu\right\} p(a, b, \tau) + sp(a, w, \tau)\right] da \\
&= -\int_{-\infty}^{\infty} \left[\{ra + w - c(a, w)\} \frac{\partial}{\partial a} p(a, w, \tau) + \left\{r - \frac{\partial}{\partial a} c(a, w)\right\} p(a, w, \tau)\right] da \\
&- \int_{-\infty}^{\infty} \left[\{ra + b - c(a, b)\} \frac{\partial}{\partial a} p(a, b, \tau) + \left\{r - \frac{\partial}{\partial a} c(a, b)\right\} p(a, b, \tau)\right] da, \qquad (75)
\end{aligned}$$

where the last equality dropped the terms $\mu p(a, b\tau)$ and $sp(a, w, \tau)$. Now integrate by parts, i.e. use $\int_{-\infty}^{\infty} u(a)\, v'(a)\, da = [u(a)\, v(a)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u'(a)\, v(a)\, da$ to find

$$\begin{aligned}&\int_{-\infty}^{\infty} \{rz + z - c(a, z)\} \frac{\partial}{\partial a} p(a, z, \tau)\, da \\ &= [\{ra + z - c(a, z)\} p(a, z, \tau)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left\{r - \frac{\partial}{\partial a} c(a, z)\right\} p(a, z, \tau)\, da.\end{aligned}$$

As the density is zero for all $a < -b/r$ and $a > a_w^*$, the first term is zero. We therefore find for the derivative of our integral

$$\begin{aligned}\frac{d}{d\tau} I(\tau) &= \int_{-\infty}^{\infty} \left\{r - \frac{\partial}{\partial a} c(a, w)\right\} p(a, w, \tau)\, da - \int_{-\infty}^{\infty} \left\{r - \frac{\partial}{\partial a} c(a, w)\right\} p(a, w, \tau)\, da \\ &+ \int_{-\infty}^{\infty} \left\{r - \frac{\partial}{\partial a} c(a, b)\right\} p(a, b, \tau)\, da - \int_{-\infty}^{\infty} \left\{r - \frac{\partial}{\partial a} c(a, b)\right\} p(a, b, \tau)\, da \\ &= 0.\end{aligned}$$

D.2 The subdistribution functions (27)

Look again at the FPEs (25). Observe that

$$\frac{d}{da} [\{ra + z - c(a, z)\} p(a, z, \tau)] = \{ra + z - c(a, z)\} \frac{\partial}{\partial a} p(a, z, \tau) + \left\{r - \frac{\partial}{\partial a} c(a, z)\right\} p(a, z, \tau)$$

and write the FPEs as

$$\begin{aligned}\frac{\partial}{\partial \tau} p(a, w, \tau) &= -\frac{d}{da} [\{ra + w - c(a, w)\} p(a, w, \tau)] \\ &\quad - sp(a, w, \tau) + \mu p(a, b, \tau), \\ \frac{\partial}{\partial \tau} p(a, b, \tau) &= -\frac{d}{da} [\{ra + b - c(a, b)\} p(a, b, \tau)] \\ &\quad + sp(a, w, \tau) - \mu p(a, b, \tau).\end{aligned} \tag{76}$$

Now consider subdistributions (as we describe subdensities), i.e. consider

$$P(a, z, \tau) \equiv \int_{-b/r}^{a} p(a, z, \tau)\, da. \tag{77}$$

As a preliminary step, compute

$$\begin{aligned}&\int_{-b/r}^{a} \frac{d}{da} [\{ra + z - c(a, z)\} p(a, z, \tau)]\, da \\ &= \{ra + z - c(a, z)\} p(a, z, \tau) - \left[\{-b + z - c(-b/r, z)\} p\left(-\frac{b}{r}, z, \tau\right)\right] \\ &= \{ra + z - c(a, z)\} p(a, z, \tau)\end{aligned} \tag{78}$$

where we used the fact (see main text) that $p\left(-\frac{b}{r}, z, \tau\right) = 0$ for all τ. Using this, we can compute the time derivative of the subdistribution (77) for $z = w$ as

$$\begin{aligned}\frac{d}{d\tau} \int_{-b/r}^{a} p(a, w, \tau)\, da &= \int_{-b/r}^{a} \frac{d}{d\tau} p(a, w, \tau)\, da \\ &= -\{ra + w - c(a, w)\} p(a, w, \tau) + \int_{-b/r}^{a} -sp(a, w, \tau) + \mu p(a, b, \tau)\, da,\end{aligned}$$

where the second equality used the PDE from (76) and the result from (78) also for $z = w$.

Using the definition from (77) and doing the same steps for $z = w$, we therefore found a differential equation system for subdistributions that reads

$$\begin{aligned}\frac{d}{d\tau}P(a,w,\tau) &= -\{ra + w - c(a,w)\}\, p(a,w,\tau) - sP(a,w,\tau) + \mu P(a,b,\tau),\\ \frac{d}{d\tau}P(a,b,\tau) &= -\{ra + b - c(a,b)\}\, p(a,b,\tau) + sP(a,w,\tau) - \mu P(a,b,\tau).\end{aligned}$$

Using the derivative of the definition of subdistributions in (77), we finally write this as

$$\begin{aligned}\frac{\partial}{\partial\tau}P(a,w,\tau) &= -\{ra + w - c(a,w)\}\,\frac{\partial}{\partial a}P(a,w,\tau) - sP(a,w,\tau) + \mu P(a,b,\tau),\\ \frac{\partial}{\partial\tau}P(a,b,\tau) &= -\{ra + b - c(a,b)\}\,\frac{\partial}{\partial a}P(a,b,\tau) - \mu P(a,b,\tau) + sP(a,w,\tau).\end{aligned}$$